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Null geodesics in the Alcubierre warp drive spacetime: the view from the bridge

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Abstract

The null geodesic equations in the Alcubierre warp drive spacetime are numerically integrated to determine the angular deflection and redshift of photons which propagate through the distortion of the "warp drive" bubble to reach an observer at the origin of the warp effect. We find that for a starship with an effective warp speed exceeding the speed of light, stars in the forward hemisphere will appear closer to the direction of motion than they would to an observer at rest. This aberration is qualitatively similar to that caused by special relativity. Behind the starship, a conical region forms from within which no signal can reach the starship, an effective "horizon". Conversely, there is also an horizon-like structure in a conical region in front of the starship, into which the starship cannot send a signal. These causal structures are somewhat analogous to the Mach cones associated with supersonic fluid flow.

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The existence of these structures suggests that the divergence of quantum vacuum energy when the starship effectively exceeds the speed of light, first discovered in two dimensions, will likely be present in four dimensions also, and prevent any warp-drive starship from ever exceeding the effective speed of light.

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Alcubierre [1] has described a spacetime which has features reminiscent of the "warp drive" common in science fiction lore. The Alcubierre solution allows a "starship" to have an apparent speed relative to distant observers which is much greater than the speed of light, an effect caused by the spacetime expanding behind and contracting in front of the starship. In such a spacetime, passengers on the starship can travel arbitrarily large distances in small amounts of proper time; further, there is no time dilation effect between the starship and clocks outside the region affected by the warp drive.

If a technology based on such a spacetime could be realized, space travel to distant points in our Universe could seem almost plausible. The Alcubierre warp drive spacetime has properties, however, which make it unlikely to be physical. As Alcubierre himself pointed out, in order to create the distortion of spacetime which produces the warp drive effect, "exotic" matter is required, which violates the weak, strong, and dominant energy conditions. Although quantized fields can locally violate the energy conditions, an analysis by Pfenning and Ford [2] has shown that the distribution of exotic matter needed to generate the warp 'bubble' around the starship appears quite implausible. Additional work has shown that any spacetime that permits *apparent* superluminal travel will inevitably violate the weak and averaged null energy conditions [3–5]. On the other hand, Van Den Broeck has recently shown how to significantly reduce the amount of negative energy density matter required for the warp drive to the order of grams [6].

Even if one could somehow obtain the negative energy density matter needed to support such a spacetime, it appears that quantized fields may prevent an Alcubierre starship from exceeding the apparent speed of light. Hiscock [8] has calculated the vacuum stress-energy tensor of a quantized scalar field in a two-dimensional reduction of the Alcubierre spacetime and shown that the stress-energy diverges if the apparent speed of the ship exceeds the speed of light. The divergence is associated with the formation of a horizon in the two-dimensional spacetime.

Despite the apparent technical difficulties involved in the construction of an Alcubierre warp drive, it is interesting to ask how the exterior Universe appears to an observer riding a starship at the center of the warp drive distortion, on the 'bridge' of the starship [1]. Our motivations for the calculation are twofold. First, it is simply a fun question to ask, and of interest to compare aberration and redshift effects to the familiar case of the view seen by a highly relativistic observer in Minkowski space. Second, owing to the existence of the two-dimensional quantum vacuum energy divergence, it is of interest to determine whether horizon-like causal structures form in four-dimensions around the starship whenever the apparent velocity exceeds the speed of light.

Another possible calculation that could be done is to study how photons propagate to observers on the far side of a warp bubble from the source. This would determine how a passing warp drive starship would affect the view of the stars seen by a distant astronomer. We have not examined this possibility here; the small angular size of the warp bubble as seen by a distant astronomer would make it unlikely that such effects would be observed, unless there are enormous numbers of warp drive starships plying the Galaxy.

The Alcubierre warp drive spacetime [1] is described by the metric

$$ds^2 = -dt^2 + \left(dx - v_s f(r_s)\ dt\right)^2 + dy^2 + dz^2 \ , \tag{1}$$

where $v_s = dx_s/dt$ is the apparent velocity of a spacetime distortion (the 'warp bubble') propagating along a trajectory described by $x_s(t)$. For simplicity we will choose the trajectory to lie along the x axis, so that $y_s(t) = z_s(t) = 0$. A radial measure of the distance from the center of the warp bubble is given by r, defined by

$$r = \left[(x - x_s)^2 + y^2 + z^2\right]^{1/2} \ . \tag{2}$$

The function $f(r)$ can be any function which is normalized to unit value at the center of the spacetime distortion, and falls off rapidly at some finite radius, asymptotically approaching zero at large r. Alcubierre gave a particular example of such a function,

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[1] One might argue that the center of the warp bubble would be in the engineering area of the starship [7]; for our purposes, we shall define the center to be the 'bridge' of the starship.

$$f(r) = \frac{tanh[\sigma(r+R)] - tanh[\sigma(r-R)]}{2tanh[\sigma R]} \quad . \tag{3}$$

This sort of function, described as a "top hat" provides a region around $r = 0$ where f is roughly constant, leading to small spacetime curvatures (and hence tidal forces) there; then a region where the function drops steeply from $f \approx 1$ to $f \approx 0$ around $r = R$. The width of the drop-off region is described by the constant σ. We shall adopt Alcubierre's choice of $f(r)$ as defining the warp drive spacetime.

The null geodesic equations are

$$p^{\alpha} p_{\beta;\alpha} = 0 \ . \tag{4}$$

Defining $dp^{\alpha}/d\lambda = p^{\beta} p^{\alpha}{}_{,\beta}$, where λ is an affine parameter measured along the null geodesic, allows one to write the geodesic equations of Eq. (4) in conventional differential form by expanding the covariant derivative, giving

$$\frac{dp^{\alpha}}{d\lambda} + \Gamma^{\alpha}{}_{\mu\nu} p^{\mu} p^{\nu} = 0 \ . \tag{5}$$

The connection coefficients, $\Gamma^{\alpha}{}_{\mu\nu}$, will introduce complicated derivatives of the functions $f(r_s)$ and v_s into the differential equations, making them analytically intractable for general initial conditions.

If the starship travels along the x-axis, the system is cylindrically symmetric about that axis. As a result, the behavior of null geodesics that reach the ship at $r = 0$ may be completely understood in terms of the subset that have the p^z component of the 4-momentum equal to zero. Only two of the geodesic equations need then be integrated, and the third nonzero component of the 4-momentum may be obtained through the null normalization condition, $p^{\alpha} p_{\alpha} = 0$. In practice, the three equations for p^t, p^x, p^y were numerically integrated, and the null normalization of the four-momentum was used as a check on the accuracy of the integration.

Evaluating the connection and using the coordinate system of Eq.(1), the (t, x, y) components of Eq.(5) take the forms:

$$\frac{dp^t}{d\lambda} + \Gamma^t{}_{tt}(p^t)^2 + \Gamma^t{}_{xx}(p^x)^2 + 2\Gamma^t{}_{tx}p^t p^x + 2\Gamma^t{}_{ty}p^t p^y + 2\Gamma^t{}_{xy}p^x p^y = 0 \ , \tag{6}$$

$$\frac{dp^x}{d\lambda} + \Gamma^x{}_{tt}(p^t)^2 + \Gamma^x{}_{xx}(p^x)^2 + 2\Gamma^x{}_{tx}p^t p^x + 2\Gamma^x{}_{ty}p^t p^y + 2\Gamma^x{}_{xy}p^x p^y = 0 \ , \tag{7}$$

$$\frac{dp^y}{d\lambda} + \Gamma^y{}_{tt}(p^t)^2 + 2\Gamma^y{}_{tx}p^t p^x = 0 \ . \tag{8}$$

In this paper we will only consider the steady-state problem where the warp speed of the starship is constant, so that $v_s = v =$ constant and $x_s = vt$.

The null geodesic equations to be integrated (Eqs.(6-8)) thus form a set of nonlinear coupled ordinary differential equations. It is also necessary to integrate the resulting expressions for p^α to obtain the trajectory of the null geodesic; only with knowledge of the trajectory can the metric connections appearing in Eqs.(6-8) be evaluated properly to complete the integration of the geodesic. Thus, Eqs.(6-8) are supplemented with the obvious relations

$$\frac{dt}{d\lambda} - p^t = 0 \ , \tag{9}$$

$$\frac{dx}{d\lambda} - p^x = 0 \ , \tag{10}$$

and

$$\frac{dy}{d\lambda} - p^y = 0 \ . \tag{11}$$

It would be inordinately difficult to fire photons from infinity and examine which ones actually pass into the (moving) warp bubble and reach an observer on the starship at $r = 0$. To avoid this problem, the null geodesic equations were numerically integrated for photon trajectories originating on the starship's bridge and propagating outward through the warp bubble to infinity. Since the effects of the warp bubble are localized, with the spacetime geometry rapidly approaching flat space at radii much greater than the bubble radius, the numerical integrations were only extended out to $r \sim 100R_{bubble}$. The resulting null geodesics were then time reversed in order to give the view as seen from the starship.

The initial conditions at the bridge ($r = 0$) consist of choosing the photon energy to be of unit value and defining the other components of the four-momentum in terms of the bridge angle of the photon via Eq.(18), so that

$$
\begin{aligned}
p^t &= 1 \ , \\
p^x &= \cos(\theta_0) + v \ , \\
p^y &= \sin(\theta_0) \ ,
\end{aligned}
\tag{12}
$$

where, again, the cylindrical symmetry about the x-axis has been used to force p^z and z to be zero always.

The two effects of particular interest in this investigation are the angular deflection of photons and the shift in their energy as they propagate into the warp distortion. The direction of photon propagation at infinity can be obtained from a simple ratio of the spatial components of the 4-momentum,

$$tan(\theta_\infty) = \frac{p^y}{p^x} \ . \tag{13}$$

To determine the angular deflection of photons, one simply compares the value of θ at infinity to the value of the equivalent angle at the center of the warp bubble. The angle observed at the bridge of the starship must be measured with reference to an orthonormal tetrad $\{e_{\hat{\mu}}\}$ moving with the starship. Such a tetrad may be defined by:

$$(e_{\hat{0}})^\alpha = u^\alpha = (1, v, 0, 0) \ , \tag{14}$$

$$(e_{\hat{1}})^\alpha = (0, 1, 0, 0) \ , \tag{15}$$

$$(e_{\hat{2}})^\alpha = (0, 0, 1, 0) \ , \tag{16}$$

$$(e_{\hat{3}})^\alpha = (0, 0, 0, 1) \ . \tag{17}$$

The angle observed for an incoming photon is then

$$\tan(\theta_0) = \frac{p^y}{p^x - vp^t} \ . \tag{18}$$

Similarly, the photon energy shift can be obtained by comparing the value of the p^t component of the photon momentum at the center of the warp bubble to the value at infinity,

$$\frac{E_0}{E_\infty} = \frac{p^t|_{r=0}}{p^t|_{r=\infty}} \ . \tag{19}$$

This measurement is not complicated by the motion of the starship, since

$$E_0 = -p^\alpha u_\alpha|_{r=0} = p^t|_{r=0} \ . \tag{20}$$

Analytic results can be obtained for special initial conditions, which will provide a useful check of the final numerically obtained general results.

Consider a null geodesic originating at $r = 0$ at $90°$ to the direction of travel. In this case it is easy to show that an exact solution of the null geodesic Eqs.(6-8) is given by

$$\begin{aligned} p^t &= 1 \ , \\ p^x &= vf \ , \\ p^y &= 1 \ . \end{aligned} \tag{21}$$

By Eq.(13), this photon trajectory is orthogonal to the direction of motion at both the bridge and infinity for all warp speeds v, and by Eq.(19), the energy of such a photon is the same at infinity and at $r = 0$, and is also independent of the warp speed.

Next, consider photons that are aligned with the direction of motion of the starship, at either $0°$ or $180°$. For such a photon, by symmetry, $p^y = p^z = 0$ always. The null geodesic equations for the remaining two components, (p^t, p^x) then simplify, becoming

$$\frac{dp^t}{d\lambda} + vf_{,x}(p^t)^2 = 0 \ , \tag{22}$$

$$\frac{dp^x}{d\lambda} + v^2 f_{,x}(p^t)^2 = 0 \ . \tag{23}$$

These equations together imply that $p^x - vp^t$ is a constant of the motion. The existence of this constant, together with the null normalization of the photon four-momentum allows the complete solution to be written in algebraic form in this case:

$$p^t = \frac{E_\infty (v \pm 1)}{v(1-f) \pm 1} \ , \tag{24}$$

$$p^x = \frac{E_\infty (v \pm 1)(vf \mp 1)}{v(1-f) \pm 1} \ , \tag{25}$$

where the upper sign refers to a photon traveling in the opposite direction to the ship, so that it meets it head-on, and the lower sign refers to a photon traveling in the same direction as the ship, chasing it from astern. Eq.(24) allows us to determine the energy of these photons as measured at $r = 0$, on the bridge, where $f = 1$. There,

$$\frac{E_0}{E_\infty} = 1 \pm v \ . \tag{26}$$

We see, as one would expect, that photons running head-on into the starship undergo a large blueshift to higher energies, while those chasing the starship from astern undergo a redshift. In fact, as the apparent velocity reaches unity, the photons directly astern are redshifted to zero energy, and never reach the starship. This is related to the existence of an event horizon in the (t, x) two dimensional spacetime when $v \geq 1$, as discussed in detail in [8].

The set of differential equations described by Eqs. (6-8) and Eqs.(9-11) were numerically integrated using a fourth-order Runge-Kutta routine with adaptive stepsize. The initial conditions were specified at the center of the warp bubble, and the null geodesic equations integrated outward. The results of these integrations were then time-reversed to determine the appearance of a distant star field to an observer riding in the center of the distortion, on the "bridge" of the "starship". The equations were integrated in 2 degree increments, for a variety of constant warp speeds v. The results are most easily interpreted in graphical form. All results illustrated here are for $\sigma = 1$; we have examined numerous other cases, and find that the qualitative behavior of the null geodesics is largely independent of the value of σ.

Figure 1 shows the angular deflection of photons which propagate into the warp bubble. The horizontal axis shows the angle of the photon's trajectory to the positive x-axis at infinity, while the vertical axis shows the angle of the photon's trajectory as measured by an observer on the bridge of the starship at $r = 0$ (the center of the warp bubble). Note that both angles are defined in terms of the apparent direction to the source of the photons (e.g., stars), not in terms of the direction of the photon's motion. For angles at infinity of less than 90 degrees, the stars' positions appear to be moved towards the direction of motion of the starship—i.e., the angle seen at the bridge is less than the angle at infinity. The magnitude of the effect grows with increasing warp speed. This clustering of apparent sources around the direction of motion is qualitatively similar to the well-known effect of the aberration of starlight in special relativity, illustrated by the two dashed curves (a particularly thorough review of the special relativistic visual effects is provided in Ref. [9]).

As the angle at infinity increases to 90 degrees and beyond, however, new effects which have no analog in special relativity are encountered. First, as shown analytically above, photons which originate precisely at 90 degrees to the direction of motion suffer no aberration whatsoever, for any warp speed. A star at 90 degrees to the direction of motion will always appear to be at 90 degrees as seen from the starship. Second, there is a conical shaped region behind the starship from which no photons reach the starship. This region first forms at warp speed $v = 1$, at which point a photon from directly behind the starship cannot catch up to the starship. As the warp speed is increased, the region behind the starship from which no photon can reach the starship grows in angular size, asymptotically approaching 90 degrees as the warp speed approaches infinity. Despite the large angular extent of this region, observers on the starship see photons originating from all angular directions, as shown in Figure 1. Starship observers do not see a large black region, since photons from smaller angles (at infinity) appear to arrive from directions further behind the ship.

This conical region from which no photon (or other signal) can reach the starship when $v > 1$ forms a sort of "horizon". There exists a similar "horizon" in front of the starship, a conical region inside of which no signal can be received *from* the starship. These horizon-like structures are the four-dimensional generalizations of the behavior previously discovered in examining the two-dimensional warp drive spacetime [8], obtained by setting $y = z = 0$.

The two-dimensional warp-drive spacetime contains event horizons surrounding the starship whenever $v \geq 1$. Figure 2 illustrates how the half-angle of the horizon-like structure grows with increasing warp speed. These structures are somewhat analogous to the familiar Mach cones associated with supersonic fluid flow.

What then would be the view from the bridge on a warp-drive starship traveling through our galaxy? The view is best understood by plotting the energy ratio of the photons versus the angle observed at the bridge, as is shown in Figure 3, while keeping in mind the aberrations illustrated in Figure 1. Assuming the warp drive speed is at least several times the speed of light, aberration would concentrate an isotropic distribution of sources along the axis of motion, the highest density of sources in a small cone directly ahead of the starship. Light from these objects would be significantly blue-shifted, most likely beyond the visible; for the photon striking the ship head-on, $E_0/E_\infty = v + 1$, as was shown analytically above. Again, photons arriving from sources perpendicular to the direction of motion, at 90 degrees, are unaffected by the warp bubble, and are observed to have $E_0/E_\infty = 1$, regardless of the value of v. Photons coming from the band between 90 degrees and the edge of the "horizon" are spread out by aberration to appear to fill the hemisphere behind the ship, with a redshift which increases without bound as the angle at the bridge approaches 180 degrees. Directly in front of the starship, blueshifting of incoming photons could create a hazard for the crew. Beginning at a warp speed of around $v = 200$, Cosmic Background photons in the forward hemisphere will be blue-shifted to energies equivalent to the solar photosphere.

More important than the view is the fact that these calculations have shown that the horizon-like structure first discovered in the two-dimensional warp drive spacetime exists in the full four-dimensional spacetime for any warp speed v that exceeds the apparent speed of light. In the two-dimensional spacetime, it was shown that the vacuum stress-energy of a quantized massless field would diverge on the horizon as it formed. Since there exist horizon-like structures in the full four-dimensional spacetime, it seems likely that a similar divergence will occur over the entire area of that structure. If so, then an infinite amount of energy would be involved in the divergence, and semiclassical backreaction effects would likely prevent the starship from ever attaining an effective warp speed greater than the speed of light. This quantum instability would prevent the operation of a warp drive vehicle even if one could overcome the significantly stringent requirements on the exotic matter needed to create such a spacetime [2,6].

It should be pointed out that while it appears plausible that the two-dimensional divergence would extend over the full horizon-like structure in four dimensions, this has not yet been demonstrated by an explicit quantum field theory calculation. Further, there remains a possibility that the quantum state of the field might be manipulated in such a way as to avoid the divergence.

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FIGURES

FIG. 1. The apparent angle to a photon source as seen by an observer on the starship bridge is compared with the angle as seen by an observer at infinity, well outside the warp bubble. Zero degrees corresponds to the direction of motion of the starship. The solid curves represent warp speeds of 0, 1, 2, 5, 10, and 100, from top to bottom in the left half of the graph. The dashed curves represent, for comparison, the special relativistic aberration for a ship traveling at speeds of 0.5, 0.9, and 0.99, from top to bottom

FIG. 2. The half-opening angle of the horizon-like "cap" behind the starship is shown as a function of the warp speed. This structure forms at warp speed $v = 1$ and grows towards a half-opening angle of 90 degrees as the warp speed diverges.

FIG. 3. The ratio of the energy of a photon as measured by an observer on the starship bridge to its energy at infinity is shown as a function of the angle to the photon source as seen by the observer on the starship bridge. Photons striking the starship head-on are blueshifted, their energy increased by a factor $v + 1$. Photons at 90 degrees are unaffected by the warp bubble. Photons coming from behind the starship are redshifted, with the redshift diverging for photons appearing to come from directly behind the ship. The solid curves represent warp speeds of 0, 1, 2, 5, and 10, from bottom to top at the left edge of the graph. For comparison, the dashed curves represent the special relativistic doppler effect for a ship traveling at speeds of 0.99, 0.9, and 0.5, from top to bottom at the left.

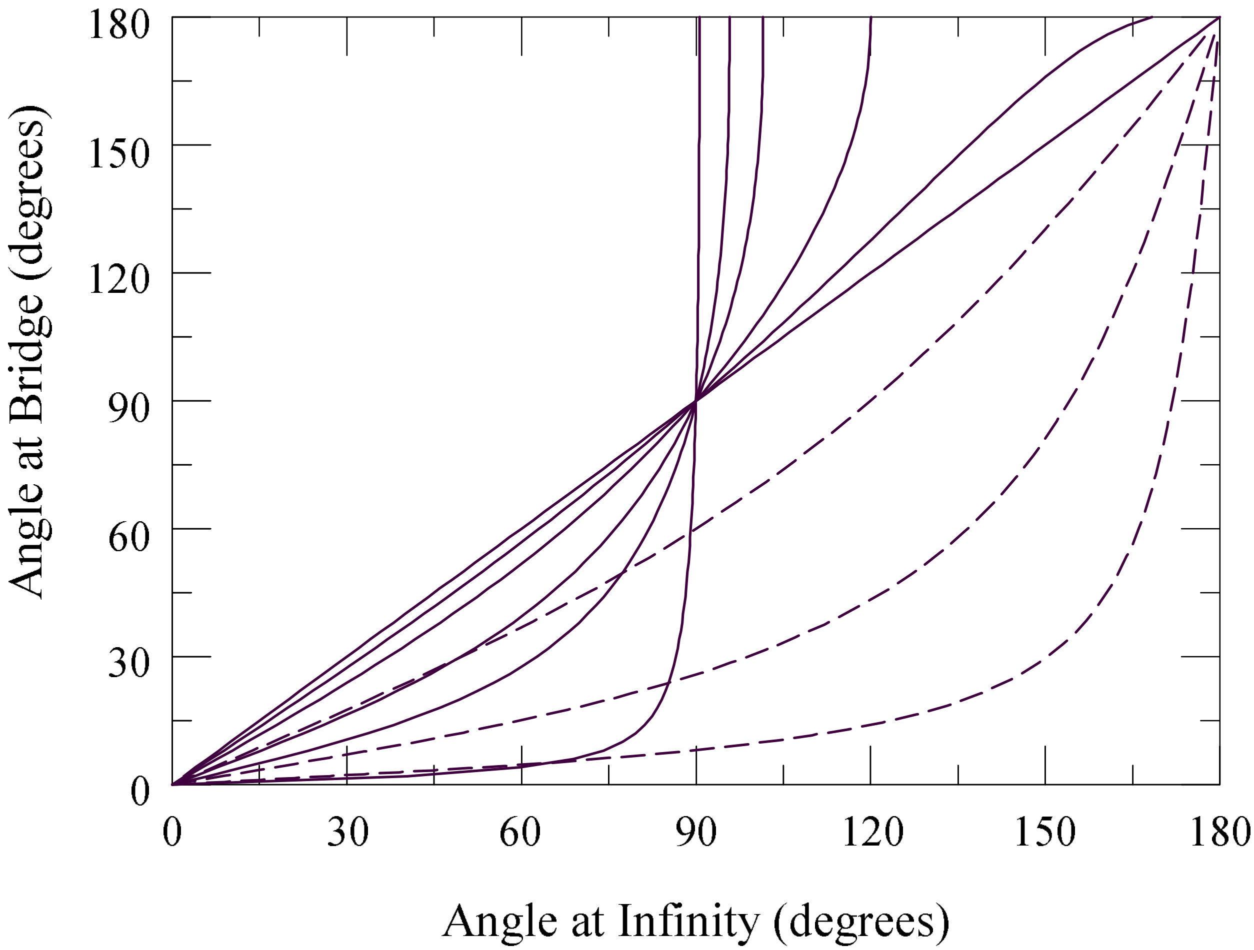
180
150
120
90
60
30
0
Angle at Bridge (degrees)
0
30
60
90
120
150
180
Angle at Infinity (degrees)

Figure 1

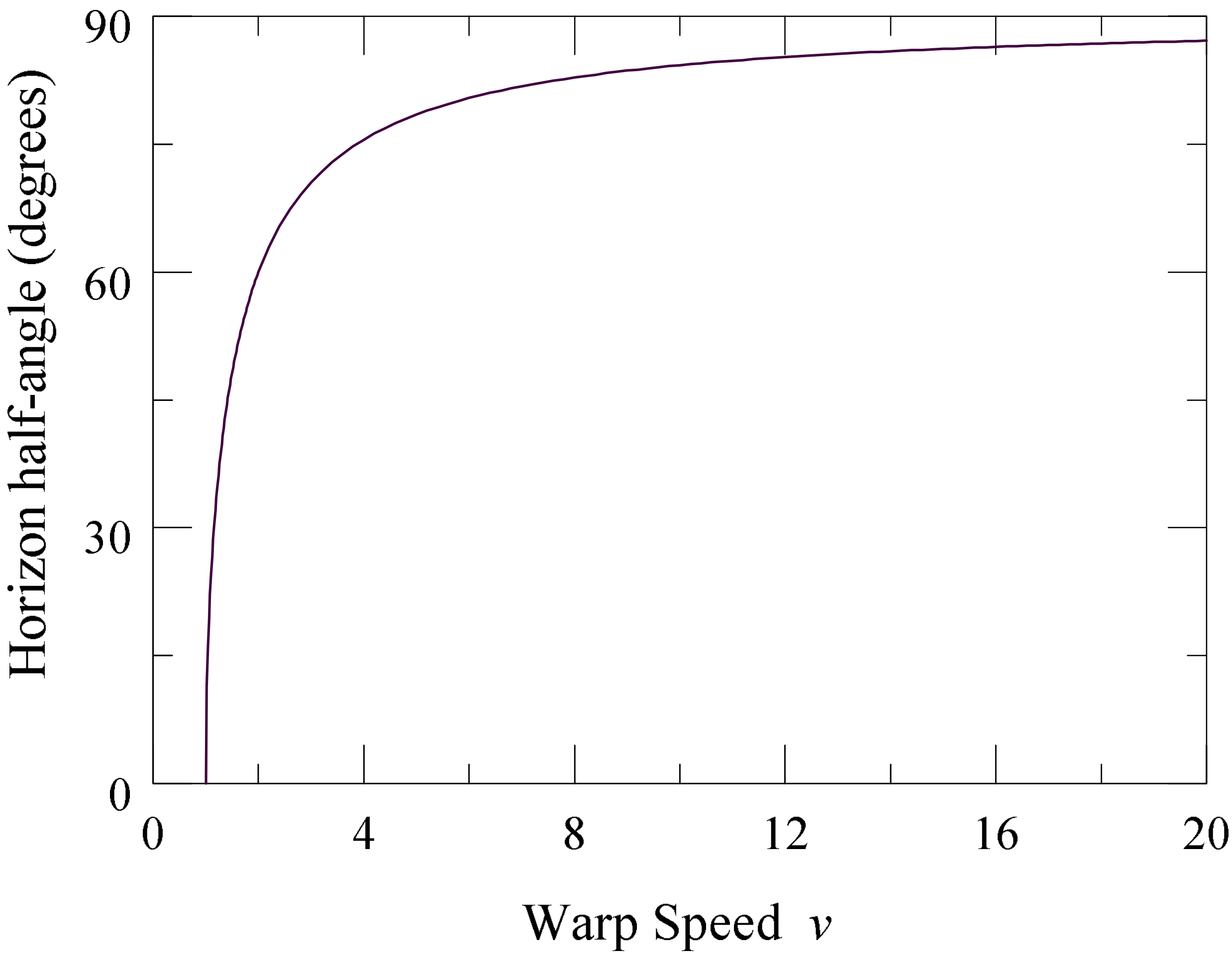

Figure 2

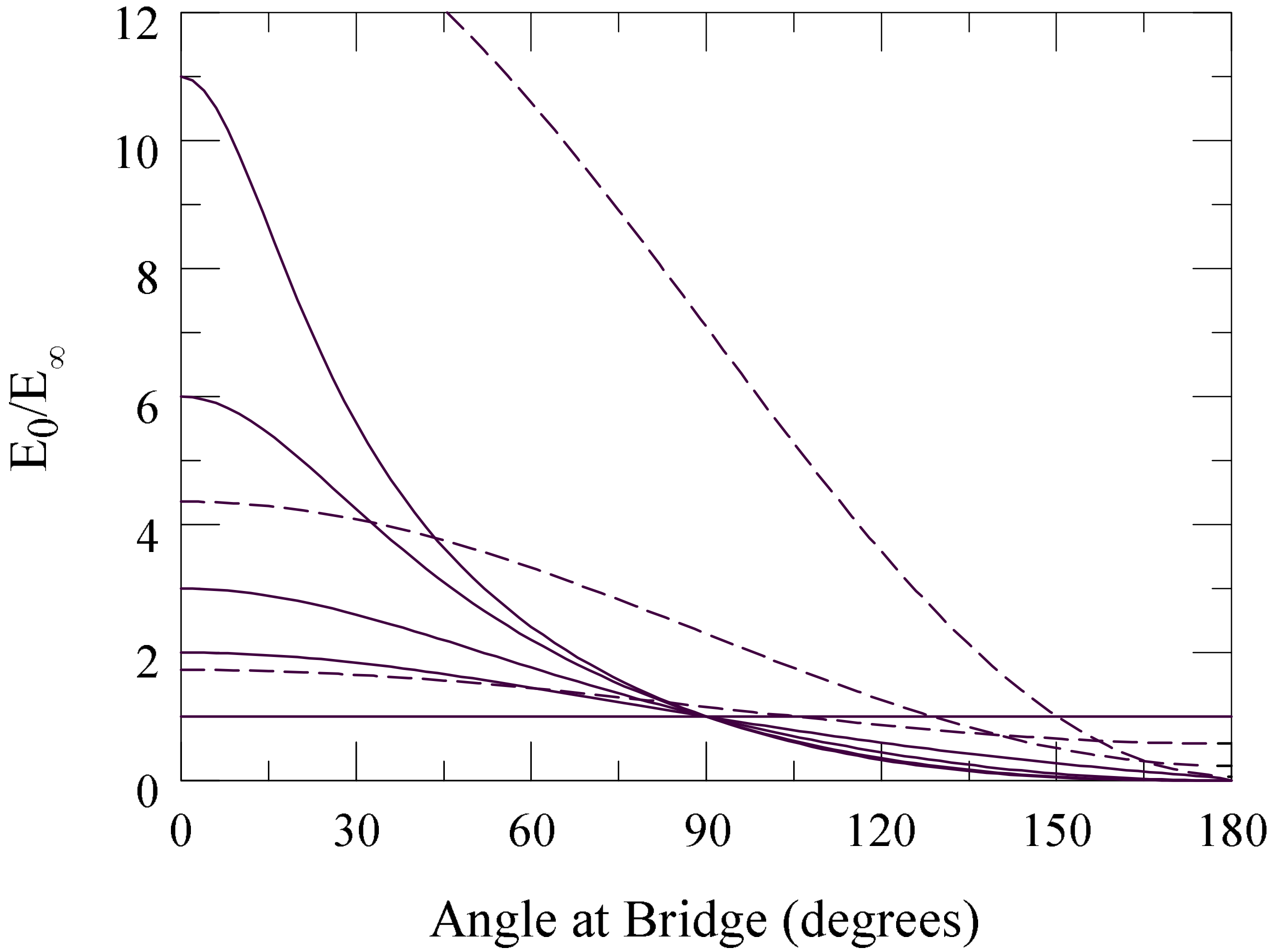
12
10
8
6
4
2
0
E_0/E_∞
0
30
60
90
120
150
180
Angle at Bridge (degrees)

Figure 3