Noisy Spherical Resonant Detector of Gravitational Waves: Veto on the Longitudinal Part of the Signal

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The equations of a resonant sphere in interaction with N secondary radial oscillators (transducers) on its surface have been found in the context of Lagrangian formalism. It has been shown the possibility to exert a veto against spurious events measuring the longitudinal component of a signal. Numerical simulations has been performed, which take into account thermal noise between resonators and the sphere surface, for a particular configuration of the transducers.

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I. INTRODUCTION

The idea of a spherical detector of gravitational waves has been considered almost since the early seventies [1]. Very accurate calculations have been performed [2] in order to use a self–gravitating object as a possible detector. Basing upon these results other authors [3] have solved the inverse problem for a free sphere in interaction with a gravitational wave. In doing so they have shown an omnidirectional sensitivity and the possibility of testing the presence of a longitudinal term in the gravitational signal. However a free spherical resonator is not a good device. In fact a set of secondary mechanical oscillators (transducers) is needed in order to convert the mechanical signal into an electromagnetic one (see [4, Thorne, p. 330]). In the spherical geometry almost 5 transducers are needed to obtain all the informations contained in the motion of its 5 quadrupolar modes. The problem of the interaction of a sphere with these secondary resonators has been solved in some particular configuration only in the nineties [5–7]. In the configurations envisaged by these authors the omnidirectional sensitivity and the possibility of total reconstruction of the signal has been shown, in this way realizing a complete observatory with only one device (cfr [8–10]). However, in the calculations performed until now, not enough attention has been paid to the possibility of vetoes against spurious events (see for instance [11]) and to the influence of thermal noise in the reconstruction of the signal. In this paper we have focus our attention upon these two problems, solving them explicitly for the dodecahedron arrangement of the transducers first described by [5]. Moreover a Lagrangian which describes the interaction of a sphere with N radial transducers on the surface has been written. This is important in order to write the most general equations involving the motion of the modes of the sphere and of the secondary oscillators. In this way it will be possible to undertake a systematic study of other transducers's configurations, enabling spherical detectors to improve their capabilities in the framework of gravitational astronomy [11,12].

II. LAGRANGIAN OF A FREE ELASTIC BODY INTERACTING WITH A GRAVITATIONAL WAVE

An elastic body in interaction with a gravitational wave could be described by ordinary elastic theory if its dimensions are very small compared to the gravitational wavelength. Gravitation acts as an external field of density force (e.g. [3])

$$f^i = \rho\, R^i = \rho\, E_{ij}(t)\, x^j \tag{2.1}$$

where E_{ij} are the components of the "electric" part of the Riemann tensor R_{i0j0}. If $\boldsymbol{u}(\boldsymbol{x}, t)$ represents the displacement from the equilibrium position in the point $\boldsymbol{x}$ then (e.g. [13])

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu\ \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu)\ \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \rho\, \boldsymbol{R} \tag{2.2}$$

Using the standard approach the solution of (2.2) could be decomposed as

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_n A_n(t) \boldsymbol{\psi}_n(\boldsymbol{x}) \tag{2.3}$$

in which $\boldsymbol{\psi}_n(\boldsymbol{x})$ are the eigenfunctions with eigenvalue ω_n of

$$\frac{\mu}{\rho}\ \boldsymbol{\nabla}^2\boldsymbol{\psi}_n(\boldsymbol{x})+\frac{(\lambda+\mu)}{\rho}\ \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_n(\boldsymbol{x}))=-\omega_n^2\boldsymbol{\psi}_n(\boldsymbol{x}) \tag{2.4}$$

The eigenfunctions $\boldsymbol{\psi}_n(\boldsymbol{x})$ are a complete set of orthonormal functions with the scalar product defined as (* means complex conjugation)

$$<\boldsymbol{g}(r,\theta,\phi)|\boldsymbol{f}(r,\theta,\phi)>=\frac{1}{M}\int\boldsymbol{g}^*\cdot\boldsymbol{f}\rho d^3x \tag{2.5}$$

where M is the mass of the elastic body. Therefore

$$\frac{1}{M}\int\boldsymbol{\psi}_n^*\cdot\boldsymbol{\psi}_{n'}\rho d^3x=\delta_{nn'} \tag{2.6}$$

Time dependent coefficients $A_n(t)$ satisfies the following forced harmonic oscillator equations

$$\ddot{A}_n+\omega_n^2\ A_n=R_n \tag{2.7}$$

where

$$R_n(t)=\frac{1}{M}\int\boldsymbol{\psi}_n^*(\boldsymbol{x})\cdot\boldsymbol{R}(\boldsymbol{x},t)\rho d^3x \tag{2.8}$$

In this way a field equation has been transformed in a set of equations for the coefficients of a function series. Now we can write the Lagrangian from which these equations are derived. This Lagrangian should be a real expression of the generalized coordinates $A_n(t)$ and $A_n^*(t)$. As far as kinetic energy of the elastic body (EB) is concerned, one has

$$T_{EB}=\frac{1}{2}\int d^3x\rho\dot{\boldsymbol{u}}^2=\frac{1}{2}M\sum_n\dot{A}_n\dot{A}_n^* \tag{2.9}$$

where last equality follows from (2.6). Potential energy of both the elastic body and the external forces are immediately written as

$$V_{EB}=\frac{1}{2}M\sum_n\omega_n^2A_nA_n^* \tag{2.10}$$

$$V_{ext}=-\frac{1}{2}M\sum_n\left[A_nR_n^*+A_n^*R_n\right] \tag{2.11}$$

III. LAGRANGIAN OF THE FREE SPHERE

If the elastic body is a sphere with radius R, then (see [2])

$$\boldsymbol{u}(r,\theta,\phi,t)=\sum_{nml}A_{nml}(t)\boldsymbol{\psi}_{nml}(r,\theta,\phi) \tag{3.1}$$

where

$$\boldsymbol{\psi}_{nlm}(r,\theta,\phi)=[a_{nl}(r)\boldsymbol{n}+b_{nl}(r)R\boldsymbol{\nabla}]Y_{lm}(\theta,\phi) \tag{3.2}$$

$$a_{nlm}\left(r\right)\ =c_{nlm}\ R\ \frac{dj_l\left(q_{nl}r\right)}{dr}+d_{nlm}\ R\ \frac{l(l+1)}{r}j_l\left(k_{nl}r\right) \tag{3.3}$$

$$b_{nlm}\left(r\right)\ =c_{nlm}j_l\left(q_{nl}r\right)+d_{nlm}\frac{d}{dr}\left(rj_l\left(k_{nl}r\right)\right) \tag{3.4}$$

and $A_{nlm}(t)=(-1)^mA_{nl-m}^*(t)$.

Now we briefly sketch the procedure used by [3] to take into account direction of the gravitational signal. In the laboratory frame O, with origin in the centre of the sphere, the gravitational wave is coming by a given direction. Let us consider a reference frame O' rotated with respect to O in such a way that z' axis coincides with the direction of propagation. All the primed quantities refer to this last system. The radial component of the surface displacement in the primed frame is then given by (see [3, eq. (7)])

$$\boldsymbol{n}' \cdot \boldsymbol{u}'(R, \theta', \phi') = \sum_{nlm'} a_{nl}(R) A'_{nlm'}(t) Y_{lm'}(\theta', \phi') \tag{3.5}$$

We are interested in the radial displacement in the laboratory frame. Let us suppose that O is rotated with respect to O' by the hour angle H (the angle between the projection of z' axis on the plane xy and x axis) and the declination δ (the angle between z' axis and the plane xy, diminished by $2\,\pi$ if obtuse) [3]. In O we have

$$\boldsymbol{n} \cdot \boldsymbol{u}(R, \theta, \phi) = \sum_{nlm} a_{nl}(R) \left(\sum_{m'} A'_{nlm'} \mathcal{D}^{(l)}_{mm'}(H, \delta) \right) Y_{lm}(\theta, \phi) \tag{3.6}$$

where $\mathcal{D}^{(l)}_{mm'}(H, \delta)$ are the rotation matrix for spherical harmonics (see e.g. [14]). Let us define

$$F_{nlm} = \sum_{m'} A'_{nlm'} \mathcal{D}^{(l)}_{mm'}(H, \delta) \tag{3.7}$$

which are the coefficients of decomposition (3.1) in O. These coefficients generalizes in a natural way coefficients F_m as defined in [3, eq. (9)], for every order in indices n and l. More precisely one has $F_m = F_{12m}$. $\mathcal{D}^{(l)}_{mm'}(H, \delta)$ are unitary matrices, therefore it is possible to invert eq. (3.7) and find $A'_{nlm'}$ as linear combinations of F_{nlm}

$$A'_{nlm'} = \sum_{m} \mathcal{D}^{(l)\,*}_{mm'}(H, \delta) F_{nlm} \tag{3.8}$$

The kinetic energy of the sphere (S) as a function of the generalized coordinates in the laboratory frame will be given by (see (2.9))

$$T_S = \frac{1}{2} M \sum_{nlm'} \dot{A}'_{nlm'} \dot{A}'^{*}_{nlm'} = \frac{1}{2} M \sum_{nlm} \dot{F}_{nlm} \dot{F}^{*}_{nlm} \tag{3.9}$$

As far as the elastic potential energy of the sphere and the potential energy of the external gravitational force are concerned one has

$$V_S = \frac{1}{2} M \sum_{nlm'} \omega^2_{nl} A'_{nlm'} A'^{*}_{nlm'} = \frac{1}{2} M \sum_{nlm} \omega^2_{nl} F_{nlm} F^{*}_{nlm} \tag{3.10}$$

$$V_{ext} = -\frac{1}{2} M \sum_{nlm'} [A'_{nlm'} R^{*}_{nlm'} + A'^{*}_{nlm'} R_{nlm'}] = -\frac{1}{2} M \sum_{nlm} [G^{*}_{nlm} F_{nlm} + G_{nlm} F^{*}_{nlm}] \tag{3.11}$$

where we have set

$$G_{nlm} = \sum_{m'} \mathcal{D}^{(l)}_{mm'} R_{nlm'} \tag{3.12}$$

which are the coefficients of the decomposition of the gravitational force in O.

IV. COUPLING WITH TRANSDUCERS

A. General Case

Let us consider a set of N harmonic mechanical oscillators with mass m_i and resonant frequency ω_i, placed on the surface of the sphere in (R, θ_i, ϕ_i). Let $q_i(t)$ be the generalized coordinates which describe radial motion of each oscillator with respect to the surface of the sphere. We make the assumption that m_i are small with respect to M, the mass of the sphere. In this way we suppose that the presence of secondary oscillators does not modify the eigenfunctions of the sphere (in the same way one suppose that the transducer of a resonant bar does not modify the eigenfunction of the cylinder). Purpose of this section is to write that part of Lagrangian relative to the motion of the resonators and to their interaction with the surface of the sphere.

Concerning the potential energy of the transducers (t), the choice of generalized coordinates brings to the expression

$$V_t = \frac{1}{2}\sum_i m_i \omega_i^2 q_i^2 \tag{4.1}$$

Moreover, if we call y_i the coordinates of the i–th oscillator in the laboratory frame, then kinetic energy of transducers is written as

$$T_t = \frac{1}{2}\sum_i m_i \dot{y}_i^2 \tag{4.2}$$

The relation between inertial and generalized coordinates is given by

$$q_i = y_i - \boldsymbol{n}\cdot\boldsymbol{u}(R,\theta_i,\phi_i,t) \overset{(3.6),(3.8)}{=} y_i - \sum_{nlm} a_{nl}(R) F_{nlm} Y_{nlm}(\theta_i,\phi_i) \tag{4.3}$$

For convenience purposes, we define, for every l, the matrix $\boldsymbol{P}^{(l)}$, whose components are

$$P^{(l)}_{mi} = Y^*_{lm}(\theta_i,\phi_i) \tag{4.4}$$

The connection between these matrices and the matrix $\boldsymbol{B}$ of ref. [5] is given by $\boldsymbol{B} = \boldsymbol{P}^{(2)*}$ With these positions the kinetic energy of the transducers is

$$\begin{aligned} T_t = \frac{1}{2}\sum_i m_i \Bigg[\dot{q}_i^2 + \dot{q}_i \sum_{nlm} a_{nl}(R)\left(P^{(l)}_{mi}{}^*\dot{F}_{nlm} + P^{(l)}_{mi}\dot{F}^*_{nlm}\right) + \\ \sum_{nlm}\sum_{n'l'm'} a_{nl}(R)a_{n'l'}(R)P^{(l)}_{mi}{}^*P^{(l')}_{m'i}\dot{F}_{nlm}\dot{F}^*_{n'l'm'}\Bigg] \end{aligned} \tag{4.5}$$

In conclusion the Lagrangian of an elastic sphere with N mechanical oscillators on its surface, undergoing an external force is

$$L = T_S + T_t - V_S - V_t - V_{ext} \tag{4.6}$$

where terms involved have been defined in (3.9), (4.5), (3.10), (4.1) and (3.11) respectively. Generalized coordinates are F_{nlm}, F^*_{nlm} and q_i, where $l = \{1,2,...\}$, $m = \{-l,-l+1,...,l\}$, and $i = \{1,2,..,N\}$. Since $F_{nl-m} = (-1)^m F^*_{nlm}$ it is sufficient to find Lagrangian equations from q_i and, say, F^*_{lm}. Therefore, to simplify the problem, we can take suitable linear combinations of the coefficients F_{nlm} in such a way to obtain real quantities. In particular, for every l it exists a unitary operator $\boldsymbol{U}^{(l)}$ which transforms irreducible tensors of rank l in real vectors having $2l+1$ components. We will use lower case letters to indicate those real vectors corresponding to irreducible tensors. One can therefore write

$$\boldsymbol{f}_{nl} = \boldsymbol{U}^{(l)}\boldsymbol{F}_{nl} \tag{4.7}$$

where $\boldsymbol{f}_{nl}$ is a vector whose components are f_{nla} ($a \in \{1,2,\ldots,2l+1\}$), while $F_{nlm}(m \in \{-l,-l+1,\ldots,l\})$ are the components of $\boldsymbol{F}_{nl}$. From now on indices a, b, c run from 1 to $2l+1$. In this way, for instance, components of $\boldsymbol{U}^{(l)}$ will be given by $U^{(l)}_{am}$. A possible choice for such unitary operators is the following

$$\begin{aligned} U^{(l)}_{1\,m} &= \delta_{m\,0} \\ U^{(l)}_{2m'\,m} &= -\frac{i}{\sqrt{2}}\left(\delta_{m'\,m} - (-1)^m\delta_{m'\,-m}\right) \\ U^{(l)}_{2m'+1\,m} &= \frac{1}{\sqrt{2}}\left(\delta_{m'\,m} + (-1)^m\delta_{m'\,-m}\right) \end{aligned} \tag{4.8}$$

where $m' \in \{1,2,\ldots,l\}$. Performing these unitary transformation Lagrangian is obtained as a function of the real and independent generalized coordinates f_{nla} and q_i. With this new set of variables, kinetic and elastic potential energy of the sphere, kinetic and potential energy of the transducers and potential energy of the external gravitational force acting on the sphere become respectively

$$T_S = \frac{1}{2} M \sum_{nla} \dot{f}_{nla}^2 \tag{4.9}$$

$$V_S = \frac{1}{2} M \sum_{nla} \omega_{nl}^2 f_{nla}^2 \tag{4.10}$$

$$\begin{aligned} T_t = \frac{1}{2} \sum_i m_i \Big[& \dot{q}_i^2 + 2\dot{q}_i \sum_{nla} a_{nl}(R) p_{ai}^{(l)} \dot{f}_{nla} + \\ & + \sum_{nla} \sum_{n'l'a'} a_{nl}(R) a_{n'l'}(R) p_{ai}^{(l)} p_{a'i}^{(l')} \dot{f}_{nla} \dot{f}_{n'l'a'} \Big] \end{aligned} \tag{4.11}$$

$$V_t = \frac{1}{2} \sum_i m_i \omega_i^2 q_i^2 \tag{4.12}$$

$$V_{ext} = -M \sum_{nla} g_{nla} f_{nla}. \tag{4.13}$$

where the matrix $\boldsymbol{p}^{(l)} = (p_{ai}^{(l)})$ is given by

$$p_{ai}^{(l)} = \sum_m U_{am}^{(l)} P_{mi}^{(l)}. \tag{4.14}$$

Obviously the potential energy of the transducers is left unchanged. Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{f}_{nla}}\right) - \frac{\partial L}{\partial f_{nla}} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

give therefore the following equations of motion for the modes of a forced elastic sphere in interaction with N harmonic mechanical oscillators

$$\begin{aligned} & M\ddot{f}_{nla} + M\omega_{nl}^2 f_{nla} - \sum_i a_{nl}(R) m_i \omega_i^2 p_{ai}^{(l)} q_i = M g_{nla} \\ & m_i \ddot{q}_i + m_i \sum_{nla} a_{nl}(r) p_{ai}^{(l)} \ddot{f}_{nla} + m_i \omega_i^2 q_i = 0 \end{aligned} \tag{4.15}$$

B. Quadrupolar Interaction

In which follows we make the simplifying assumptions that secondary oscillators are identical, their common frequency is equal to ω_{12} (the frequency of the mode $n = 1,\ l = 2$ of the sphere), their mass is m and the external force acting on the sphere has only $l = 2$ components. Moreover, for high quality factors (τ_{nlm} is the relaxation time of the nlm mode)

$$|\omega_{12} - \omega_{nl}| >> \frac{1}{\tau} \qquad \tau = min\{\tau_{12m}, \tau_{nlm'}\}; \qquad \begin{matrix} n \neq 1 \\ l \neq 2 \end{matrix}$$

therefore if the motion of the transducers is studied in the frequency domain near the frequency ω_{12}, only modes with $n = 1$ and $l = 2$ must be taken into account. To simplify notations we set

$$p_{ai}^{(2)} \equiv p_{ai} \qquad f_{12a} \equiv f_a \qquad g_{12a} \equiv g_a \qquad \omega_{12} = \omega_0 \qquad a_{12}(R) \equiv \alpha$$

With these assumptions we have

$$T_S = \frac{1}{2} M \sum_a \dot{f}_a^2 \tag{4.16}$$

$$V_S = \frac{1}{2} M\omega_0^2 \sum_a f_a^2 \tag{4.17}$$

$$T_t = \frac{m}{2} \sum_i \left[\dot{q}_i^2 + 2\dot{q}_i \sum_a \alpha p_{ai} \dot{f}_a + \sum_{aa'} \alpha^2 p_{ai} p_{a'i} \dot{f}_a \dot{f}_{a'} \right] \tag{4.18}$$

$$V_t = \frac{m\omega^2}{2} \sum_i q_i^2 \tag{4.19}$$

$$V_{ext} = -\frac{1}{2} M \sum_a g_a f_a \tag{4.20}$$

by means of which the following equations of motions are obtained (^{t} means transpose)

$$\begin{pmatrix} M\mathcal{I}_5 & 0 \\ m\alpha \boldsymbol{p}^t & m\boldsymbol{\mathcal{I}}_N \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{f}} \\ \ddot{\boldsymbol{q}} \end{pmatrix} + \begin{pmatrix} M\omega_0^2 \mathcal{I}_5 & -m\omega_0^2 \alpha \boldsymbol{p} \\ 0 & m\omega_0^2 \mathcal{I}_N \end{pmatrix} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{I}}_5 & 0 \\ 0 & \boldsymbol{\mathcal{I}}_N \end{pmatrix} \begin{pmatrix} M\boldsymbol{g} \\ 0 \end{pmatrix} \tag{4.21}$$

where $\boldsymbol{\mathcal{I}}_n$ is the identity matrix in n–dimensional space.

If there are forces $r_i^{(ns)}$ between oscillators and the surface of the sphere (for instance thermal noise forces), then a new potential energy term must be added to the Lagrangian

$$W_{ns} = -\sum_i q_i r_i^{(ns)} \tag{4.22}$$

The equations of motion then become (cfr [7, eq. (25)])

$$\begin{pmatrix} M\boldsymbol{\mathcal{I}}_5 & 0 \\ m\alpha \boldsymbol{p}^t & m\boldsymbol{\mathcal{I}}_N \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{f}} \\ \ddot{\boldsymbol{q}} \end{pmatrix} + \begin{pmatrix} M\omega_0^2 \boldsymbol{\mathcal{I}}_5 & -m\omega_0^2 \alpha \boldsymbol{p} \\ 0 & m\omega_0^2 \boldsymbol{\mathcal{I}}_N \end{pmatrix} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{I}}_5 & -\alpha \boldsymbol{p} \\ 0 & \boldsymbol{\mathcal{I}}_N \end{pmatrix} \begin{pmatrix} M\boldsymbol{g} \\ \boldsymbol{r}^{(ns)} \end{pmatrix} \tag{4.23}$$

The coefficient $\alpha = a_{12}(R)$ can be calculated from (3.3) solving eigenfunction equations (2.4). One find that $\alpha = -2.886$.

Now the system of $5+N$ coupled oscillators must be decoupled thus finding the eigenfrequencies. It is important to check if the frequency spread due to the decoupling is consistent with the assumption that only modes with $n = 1$ and $l = 2$ have been taken into account. Calling $\tilde{\omega}_a$ these eigenfrequencies and $\tilde{\tau}_a$ the relaxation times of the eigenmodes, then the following consistency relation should hold

$$|\tilde{\omega}_a - \omega_{nl}| >> \frac{1}{\tau} \qquad \tau = min\{\tilde{\tau}_a, \tau_{nlm}\} \qquad \begin{matrix} n \neq 1 \\ l \neq 2 \end{matrix}$$

As it will be seen, the above condition is fulfilled for high quality factors and $m_i/M << 1$. To perform such a program a particular arrangement of transducers must be chosen. It is what we are going to do in the next section.

V. DODECAHEDRON ARRANGEMENT

Now, the problem of the decoupling of the system made by the quadrupolar modes of the sphere interacting with N harmonic oscillators is to be solved. In order to do so the number and exact position of the mechanical oscillators on the surface must be known. This choice is of fundamental importance to keep isotropy in the sensitivity. Until now, two arrangements in particular are studied in view of an experimental realization. The first one [5,7] is obtained placing six radial resonators along the directions of a dodecahedron (this arrangement was yet investigated [8–10] in the framework of the study of a network of resonant gravitational antennae). In the second one [6] 5 transducers (four of which tangential) are placed in such a way that they could be excited by only one mode of the sphere; in this way decoupling is realized at once. In this paper we restrict ourselves to the first case.

Let us consider therefore the dodecahedron arrangement. Five transducers are placed on the same parallel of the sphere. Their common azimuth angle is $\theta = \arcsin(2/\sqrt{5})$; their polar angles are given by $\phi_k = 2k\pi/5$ ($k \in \{1, \ldots, 5\}$). Last transducer is placed at a pole. In this case matrix $\boldsymbol{p}$ become

$$\boldsymbol{p} = \sqrt{\frac{3}{5\pi}} \begin{pmatrix} 1 & c_4 & c_2 & c_2 & c_4 & 0 \\ 0 & -s_4 & s_2 & -s_2 & s_4 & 0 \\ -1 & -c_2 & -c_4 & -c_4 & -c_2 & 0 \\ 0 & s_2 & s_4 & -s_4 & -s_2 & 0 \\ -a_1 & -a_1 & -a_1 & -a_1 & -a_1 & 5a_1 \end{pmatrix} \qquad \begin{array}{c} c_2 = \cos\left(\frac{2\pi}{5}\right) \\ s_2 = \sin\left(\frac{2\pi}{5}\right) \\ a_1 = \frac{1}{2\sqrt{3}} \end{array} \qquad \begin{array}{c} c_4 = \cos\left(\frac{4\pi}{5}\right) \\ s_4 = \sin\left(\frac{4\pi}{5}\right) \end{array} \tag{5.1}$$

Such a matrix has the following important properties (cfr [5,7])

$$\boldsymbol{p}\boldsymbol{p}^t = \frac{3}{2\pi}\boldsymbol{\mathcal{I}}_5; \qquad \sum_i p_{ai} = 0 \tag{5.2}$$

Decoupling system (4.23) means solving an eigenvalue problem. The eigenfunctions are the so called normal coordinates, while the eigenfrequencies are called frequencies of free vibration (see [15]). In the case of this arrangement, normal coordinates are divided in three set: two quintuplet with eigenfrequencies $\omega_+ = \omega_0\,\lambda_+$ and $\omega_- = \omega_0\,\lambda_-$ and a singlet with frequency ω_0 where

$$\lambda_\pm = 1 \pm \frac{1}{2}\sqrt{\frac{3\alpha^2}{2\pi}\mu} \qquad \mu = \frac{m}{M} \tag{5.3}$$

The normal coordinates ζ^p $(p = 1, \ldots, 11)$ are therefore naturally divided in three groups: $\boldsymbol{\zeta}_- = (\zeta_p)$ $p \in \{1, \ldots, 5\}$, $\zeta = \zeta_6$ and $\boldsymbol{\zeta}_+ = (\zeta_i)$ $p \in \{7, \ldots, 11\}$. The equations of forced motion are (detailed calculations are found in appendix A)

$$\begin{aligned} \ddot{\boldsymbol{\zeta}}_- + \lambda_-^2\omega_0^2\boldsymbol{\zeta}_- &= \sqrt{\frac{M}{2}}\lambda_-\boldsymbol{g} \\ \ddot{\zeta} + \omega_0^2\zeta &= 0 \\ \ddot{\boldsymbol{\zeta}}_+ + \lambda_+^2\omega_0^2\boldsymbol{\zeta}_+ &= \sqrt{\frac{M}{2}}\lambda_+\boldsymbol{g} \end{aligned} \tag{5.4}$$

Therefore if $\boldsymbol{g}$ is given, one can find the motion of coefficients f_a and transducers q_i. In particular because of the form of matrix $\boldsymbol{A}$ (see eq. (A9)) we have

$$\begin{aligned} \boldsymbol{f} &= \sqrt{\frac{1}{2M}}\left(\lambda_-\boldsymbol{\zeta}_- + \lambda_+\boldsymbol{\zeta}_+\right) \\ \boldsymbol{q} &= \sqrt{\frac{\pi}{3m}}\boldsymbol{p}^t\left(\lambda_-\boldsymbol{\zeta}_- - \lambda_+\boldsymbol{\zeta}_+\right) + \frac{\zeta}{\sqrt{6m}}\mathbf{1}_{(6\times 1)} \end{aligned} \tag{5.5}$$

VI. DECOMPOSITION OF THE GRAVITATIONAL WAVE FORCE

A gravitational wave, in the framework of a general metric theory of gravity, is described as a combination of six matrices. A possible choice is the following (see also [16]):

$$\boldsymbol{E}^{(S)} = \frac{2}{\sqrt{15}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

$$\boldsymbol{E}^{(0)} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \boldsymbol{E}^{(\pm1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm i \\ 1 & \pm i & 0 \end{pmatrix} \qquad \boldsymbol{E}^{(\pm2)} = \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.2}$$

Matrix (6.1) accounts for the scalar part of the signal, while the others describe the quadrupolar components of the wave. They are relative to $m = 0$, $m = \pm1$ and $m = \pm2$ respectively. One may argue that, once the direction of propagation is known, a sphere with dodecahedron arrangement could detect, at least in principle, all six components. However, the scalar mode of the sphere has a principal resonant frequency ω_{10}, which is a little more than twice the quadrupolar one ω_{12} (e.g. [16]). Current experimental schemes needs that, in order to measure the scalar mode of the sphere, this must be coupled with a secondary oscillator having ω_{10} as a resonant frequency. Therefore, being instead ω_{12} the resonant frequency of each transducer, scalar part of the signal cannot be detected. Besides, propagation direction is in general unknown. One is led to the conclusion that with a dodecahedron arrangement it is possible to get the direction of propagation and three components of the five quadrupolar possible ones. In the framework of general relativity, a gravitational wave could only have two possible states of polarization, the $\boldsymbol{E}^{(\pm 2)}$ ones. Therefore one can think that a sphere could be used to test general relativity. However, the discovery of PSR1913+16 [17] and 20 years of subsequent observations have put strong experimental limitations on other theory of gravitation different from general relativity. This last one passed all these new experimental tests with complete success [18] showing that "the correct theory of gravity must make predictions that are asymptotically close to those of general relativity over a vast range of classical circumstances" [19]. Therefore another possibility is to consider general relativity as the correct metric theory of gravity and then to use the degree of freedom left as a veto against spurious signals. This is the strategy of the present work. In which follows we consider a gravitational wave in O' having the form

$$\boldsymbol{E}'(t) = E_R(t)\boldsymbol{E}^{(+2)} + E_L(t)\boldsymbol{E}^{(-2)} + E_l(t)\boldsymbol{E}^{(0)} \tag{6.3}$$

where $E_L = E_R^*$ and the connection with the usual polarization amplitudes is $E_+(t) = E_R(t) + E_L(t)$ and $E_\times(t) = i\left(E_R(t) - E_L(t)\right)$, while E_l is the eventual longitudinal polarization.

With this position it is possible to find, for every n, the vector $\boldsymbol{g}_n$ (whose components are g_{n2a}) which enters in the equations of motions of the normal modes (5.4) as a function of $\boldsymbol{E}'(t)$ by means of eqs. (4.7), (3.12), (2.8) and (2.1). Detailed calculations are found in appendix B. Here we give only the final result

$$\begin{aligned}
(g_n)_5(t) &= \frac{\gamma_n}{\sqrt{2}}\left[\frac{1}{4}(3-\cos 2\delta)\cos 2H\ E_+ - \sin\delta\sin 2H\ E_\times + \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{\sqrt{3}}{4}(1+\cos 2\delta)\cos 2H\ E_l\right] \\
(g_n)_4(t) &= \frac{\gamma_n}{\sqrt{2}}\left[\frac{1}{4}(3-\cos 2\delta)\sin 2H\ E_+ + \sin\delta\cos 2H\ E_\times + \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{\sqrt{3}}{4}(1+\cos 2\delta)\sin 2H\ E_l\right] \\
(g_n)_3(t) &= \frac{\gamma_n}{\sqrt{2}}\left(\frac{1}{2}\sin 2\delta\cos H\ E_+ - \cos\delta\sin H E_\times - \frac{\sqrt{3}}{2}\sin 2\delta\cos H\ E_l\right) \\
(g_n)_2(t) &= \frac{\gamma_n}{\sqrt{2}}\left(\frac{1}{2}\sin 2\delta\sin H\ E_+ + \cos\delta\cos H E_\times - \frac{\sqrt{3}}{2}\sin 2\delta\sin H\ E_l\right) \\
(g_n)_1(t) &= \frac{\gamma_n}{\sqrt{2}}\left[\frac{\sqrt{3}}{4}(1+\cos 2\delta)E_+ + \frac{1}{4}(1-3\cos 2\delta)E_l\right]
\end{aligned} \tag{6.4}$$

where γ_n is defined in appendix B.

It has been therefore explicitly found the motion of mechanical transducers on the surface of the sphere as a function of direction and polarization amplitudes of a signal (by means of (5.5), (5.4) and (6.4)), solving the direct problem of motion.

VII. INVERSE PROBLEM

Inverse problem lies in the determination of the incoming signal characteristics for a given response of the detection apparatus. In this case what is measured is the 6 component vector $\boldsymbol{q}(t)$, that is the motion of transducers with respect to the surface of the sphere. From this vector one should obtain $\boldsymbol{\zeta}(t)$, describing the motion of normal modes, and $\boldsymbol{g}$, the generalized force for the normal coordinates. Finally, inverting system (6.4) one should get those quantities which describe the signal, that is direction, polarizations E_+ and $E_\times$ and an eventual polarization with $m = \pm 1$ or $m = 0$. In which follows we consider only the possibility that the signal could have a longitudinal polarization E_l (see (6.3)).

In Fourier space ($f(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} f(t)\exp(-i\omega t)\ dt$) using properties (5.2) of matrix $\boldsymbol{p}$ and eq. (5.4) one gets

$$\boldsymbol{g}(\omega) = -\frac{2\pi}{3\alpha} \frac{\left(\omega_0^2 - \omega^2\right)^2}{\omega^2} \boldsymbol{p}^t \boldsymbol{q}(\omega) \tag{7.1}$$

This last equation implies, together with (5.2)

$$\boldsymbol{q}^t(\omega)\boldsymbol{q}(\omega) \propto \boldsymbol{g}^t(\omega)\boldsymbol{g}(\omega) = \frac{\gamma_1^2}{2}\left(E_+^2 + E_\times^2 + E_l^2\right) \tag{7.2}$$

where last equality holds taking into account eqs. (4.7), (3.12) and (B1). Therefore the sum of the squares of the detector responses are independent on the direction. This means that this transducer arrangement gives isotropic sensitivity. The result could be also obtained calculating directly the sum of the squares of the components of $\boldsymbol{g}$ in (6.4).

Inverse problem is then solved when system (6.4) is inverted. To this aim it is useful to define an amplitude vector $\boldsymbol{a}$

$$\boldsymbol{a} = \frac{\sqrt{2}}{\gamma_1}\boldsymbol{g}, \qquad \boldsymbol{a}^2 = E_+^2 + E_\times^2 + E_l^2, \tag{7.3}$$

whose length is the sum of the squares of the three polarization amplitudes of the signal. Performing this program (results are found in appendix C) one obtains a six-th degree algebraic equation for $x = \tan H/2$ with coefficients depending on $\boldsymbol{a}$. Then a relation for $\tan\delta$ as a function of H and $\boldsymbol{a}$; finally three relations for E_+, $E_\times$, E_l as functions of H, δ and $\boldsymbol{a}$.

As it will be seen better in the following, since for a gravitational signal E_l must vanish, only two of the six solutions for H can be accepted. In fact longitudinal components will be zero for only two values of H. These acceptable solutions differ only because they describe signals coming along opposite directions. In fact the difference between their hour angles is π. This property holds also for the discarded solutions: they are in pairs differing only for the opposite direction of propagation.

Solution of the inverse problem lies therefore in the solution of the equation for H. This could be done numerically once vector $\boldsymbol{a}$ is known. To this purpose a code has been developed which solves the direct and the inverse problem. The obtained calculation precision on the reconstruction of the signal is very much higher than the errors introduced by thermal noise which we are going to consider in next section.

VIII. THERMAL NOISE

So far the theory dealt with ideal detectors, that is to say noiseless ones. However, as it is well known, real detectors are not at rest before the interaction with a signal but, because of the heat bath in which they are thermodynamically in equilibrium, they have a Brownian motion which give them a certain amount of energy. For this reason a detector (a part from the quantistic aspects of the problem, aspects which come out indeed when the temperatures are lower than those actually involved at the present) could only reveal signals yielding an amount of energy greater or at least comparable to the thermic one. Therefore, for any given temperature, there will be a lowest signal detectable. In this section we analyze how the behaviour of the system described until now is to be modified by the presence of thermal noise.

To this aim, the problems that must be solved are at least two. The first one is to understand which is the noise that one could expect in the transducers on the surface of the sphere at a certain temperature. The second one concerns the knowledge of the decreasing in the precision with which not only the signal could be deconvolved (by means of the determination of the arrival direction and polarization amplitudes) but also with which could be used the veto on the longitudinal part of the signal. In other words a confidence level must be decided for which a certain value of E_l (which could not be exactly zero) is to be considered equal to zero or not.

Let us begin to consider the first problem answering to the question: which noise is to be expected on the transducers at a given temperature? As a first estimate to be compared with the result of a more accurate calculation we may suppose, if the sphere mass is very much greater than that of the oscillators, that the Brownian noise cause a motion whose amplitude with respect the surface of the sphere is of the order of

$$q_i^{(0)} = \sqrt{\frac{2\,k_B\,T}{m\,\omega_0^2}} \tag{8.1}$$

This result shall be valid as an order of magnitude, allowing settlements which, in general, depend upon the position of the oscillators on the surface of the sphere. We will see that, in the dodecahedron configuration this result is indeed correct being the same for every transducer.

Strictly speaking, the equipartition energy theorem, that we used for giving the order of magnitude of the amplitudes $q_i^{(0)}$, is valid, in classical mechanics, only for every independent quadratic term in the energy and for every one of these the square–mean value is $(1/2)\,k_B\,T$. Therefore, when we consider interacting oscillators we can apply it just to the normal coordinates. It must be observed, however, that the theory developed until now is only an approximation: in fact together with the lagrangian of the elastic body in interaction with the secondary oscillators we have not consider the dissipation function [15, §1–5]. As it is well known it is a quadratic function of the velocities $\mathcal{F} = (1/2)\mathcal{F}_{ij}\dot{\eta}^i\dot{\eta}^j$ and $2\mathcal{F}$ is the rate of energy dissipation due to friction [15, §2.6]. The motion equations become [15, §6.5]

$$T_{ij}\ddot{\eta}^j + \mathcal{F}_{ij}\dot{\eta}^j + V_{ij}\eta^j = 0$$

It is not in general possible to find normal modes for any arbitrary dissipation function. In other words it is not possible to decompose the system in non–interacting modes diagonalizing simultaneously the three matrices T_{ij}, V_{ij} and $\mathcal{F}_{ij}$. Therefore the equipartition theorem could not be applied even to the normal coordinates. If we make the assumption of high quality factor we can suppose that non–diagonal quadratic terms are very small with respect to the others so we can safely neglect them. In the framework of this approximation we can therefore apply the equipartition theorem to the normal coordinates, whose Brownian motion will then be given by (see (5.4))

$$\begin{aligned} \boldsymbol{\zeta}_- &= \frac{\sqrt{2\,k\,T}}{\lambda_-\,\omega_0}\boldsymbol{n}_-(t) \\ \zeta &= \frac{\sqrt{2\,k\,T}}{\omega_0}n(t) \\ \boldsymbol{\zeta}_+ &= \frac{\sqrt{2\,k\,T}}{\lambda_+\,\omega_0}\boldsymbol{n}_+(t) \end{aligned} \tag{8.2}$$

where $\boldsymbol{n}_-(t)$, $n(t)$ and $\boldsymbol{n}_+(t)$ are eleven independent adimensional random functions (where "random" means that their average values vanish) whose root–mean–square deviations are equal to one. The relative coordinates $\boldsymbol{q}(t)$ are given as a function of the normal ones $\boldsymbol{\zeta}(t)$ by the equation (5.5) which now could be written as

$$\boldsymbol{q} = \frac{\sqrt{2\,k\,T}}{\omega_0}\left[\sqrt{\frac{\pi}{3m}}\boldsymbol{p}^t\ (n_-(t) - n_+(t)) + \frac{n(t)}{\sqrt{6m}}\mathbf{1}_{(6\times1)}\right] \tag{8.3}$$

Let us now define the following five functions

$$\boldsymbol{v}(t) = \frac{1}{\sqrt{2}}\,(n_-(t) - n_+(t)) \tag{8.4}$$

For these functions (from now on $< f >$ means the average of a given quantity $f(t)$) one obviously has that $< \boldsymbol{v} >= 0$ (that is to say they are random functions) and $< v_a^2 >= 1$. We can therefore write

$$\boldsymbol{q}(t) = \frac{1}{\omega_0}\sqrt{\frac{2\,k\,T}{3\,m}}\left[\sqrt{2\,\pi}\boldsymbol{p}^t\,\boldsymbol{v}(t) + \frac{1}{\sqrt{2}}\,n(t)\,\mathbf{1}_{(6\times1)}\right] \tag{8.5}$$

Let now be $\boldsymbol{w}(t)$ six functions defined as

$$\boldsymbol{w}(t) = \sqrt{\frac{2}{5}}\sqrt{2\pi}\boldsymbol{p}^t\boldsymbol{v}(t) \tag{8.6}$$

As it is easily seen these are random functions and we have also that $< w_i^2 >= 1$. This follows directly from the fact that $(\boldsymbol{p}^t\,\boldsymbol{p})_{ii} = 5/(4\pi)$. This property of the matrix $\boldsymbol{p}$ is of fundamental importance in order that all the transducers have the same noise amplitude. Introducing these new functions we obtain

$$\boldsymbol{q}(t) = \frac{1}{\omega_0}\sqrt{\frac{2\,k\,T}{3\,m}}\left[\sqrt{\frac{5}{2}}\boldsymbol{w}(t) + \sqrt{\frac{1}{2}}\,n(t)\,\mathbf{1}_{(6\times1)}\right] \tag{8.7}$$

putting now

$$\boldsymbol{z}(t) = \sqrt{\frac{1}{3}}\left(\sqrt{\frac{5}{2}}\boldsymbol{w}(t) + \sqrt{\frac{1}{2}}\,n(t)\,\mathbf{1}_{(6\times1)}\right) \tag{8.8}$$

where once again $< \boldsymbol{z} >= 0$ and $< z_i^2 >= 1$, one gets

$$\boldsymbol{q}(t) = \frac{1}{\omega_0}\sqrt{\frac{2\,k\,T}{m}}\boldsymbol{z}(t) \tag{8.9}$$

We have therefore shown that the fluctuations due to the Brownian motion give rise to a random motion of the coordinates $\boldsymbol{q}(t)$ with root–mean–value just given by (8.1). It is to be remarked that this result depends on the particular arrangement of the secondary oscillators on the surface of the sphere. A priori one would in fact expect a position dependent result.

We have therefore given the answer to the first question put by the presence of noise. The next section is devoted to the solution of the second problem envisaged here: the reconstruction precision.

IX. NUMERICAL SOLUTION

In this section we are interested in the answer to the problem of knowing the decrease in the signal reconstruction precision when brownian motion is to take into account. It is to be remarked that this precision decrease concerns not only the arrival direction and the transverse polarization amplitudes, but also the value of the longitudinal component which, in case of gravitational event, must be zero but, because of noise, could not be "mathematically" so. Therefore we have to precise the criterion with which this signal component is to be considered "physically" vanishing. Because of the complexity of the system we can not simply apply a kind of analysis based on the error propagation: the best solution is to perform numerical simulations.

The strategy is to write a code which, once the signal characteristics are given, calculates the components of the vector $\boldsymbol{a}$ (that is to say the response of the device), gives them a gaussian noise with null mean value and unitary root–mean-square deviation and finally solves the inverse problem. When this is done six solutions are found which represents a possible signal. The four solutions which have the greater longitudinal polarization are discarded. With an iterative procedure over n attempts ($n >> 1$), one could extract two kinds of information (for every value of the signal–to–noise ratio): the first one is about the error on the reconstruction of the initial signal; the second one is about the typical value of the ratio between the longitudinal component E_l and the amplitude $E = (E_+^2 + E_\times^2)^{(1/2)}$ which one could expect from a real measure in the case of gravitational (that is to say transverse) signal.

We have reached the conclusion that the signal could be reconstructed when the signal–to–noise ratio (SNR) is equal or greater than 10. If this is the case the relative error on the amplitude E and the absolute one on the δ angle is of the order of $1/SNR$. A great lack of precision appears on the H angle and on the polarization angle $\psi = \arctan E_\times/E_+$ (that is to say on the knowledge of the two polarization amplitude) when δ assumes values closer to $\pm\pi/2$, as it was anyway well-founded to be expected. As far as the ratio E_l/E is concerned, it is distributed around 0 with a typical spread which is of the order of the reciprocal of SNR. From this analysis one could conclude that, if the response of the detector lead to a reconstruction of the signal in which the six solutions are such that the ratio E_l/E is significantly greater than $1/SNR$, then this signal is certainly to be considered a spurious event and rejected as a possible gravitational event.

In tables below we report some of the results and graphics concerning two simulations over 1000 attempts. In the first case the signal was supposed to come from a source placed at hour angle $H = 1.0\,rad$ and declination $\delta = 0.7\,rad$. The amplitude E of the wave is 10 (in units of the SNR), with the two polarization amplitudes given by $E_+ = 8$ and $E_\times = 6$. In the second case the source is placed at hour angle $H = 1.0\,rad$ and declination $\delta = 1.3\,rad$. The signal amplitude is still given by $E = 10$, with $E_+ = 6$ and $E_\times = 8$. The results of the simulations are given in table I. The quantity $\Delta\Omega$ is the error on the solid angle; one has $\Delta\Omega = \Delta H\,\Delta\delta\,\cos\delta$. In Figs. 1 and 2 are shown the scattered plots of reconstruction position in the two cases.

An analysis of the results shows immediately that the precision on H decrease when the source become nearer to the poles. However this have not substantially repercussions on the solid angle precision. Joined directly with the "spreading" of H is the uncertainty on the polarization amplitudes which increases when the signal direction approaches a polar one.

A consequence of this analysis is that, once a reference frame O is chosen, the isotropy of the sphere with thermal noise concerns direction, total amplitude E and amplitude of the longitudinal component E_l of the signal. As far as the possibility of reconstruction of the polarization amplitudes E_+ and $E_\times$, it is strongly dependent on the declination angle. That is, sphere isotropy on the reconstruction of polarization angle $\psi = \arctan(E_\times/E_+)$ is broken by thermal noise near polar singularity.

X. CONCLUSIONS

An elastic sphere could represent, in a no longer far future, an important device in the context of gravitational astronomy. This lies in its peculiarity of total reconstruction of the signal with isotropic sensitivity and in the possibility of testing the transversality of the Riemann tensor. This last fact is very important because it allows to exert a veto against spurious signals. We think that the operation usefulness of a gravitational antenna does not depend only on its sensitivity or capability of signal reconstruction, but also on the confidence level with which an observed event could be associated to a gravitational wave (see [8–10]). The Lagrangian approach we have used, has been developed in the context of this strategy in order to write the correct equations of the coupled oscillators using the simpler formalism. In fact we think that a serious study should be done in order to find other configurations of the secondary oscillators for which also an eventual scalar amplitude of the signal could be measured and used as a veto. A first approach in this direction has been attempted with a different method by [11].

The analysis of the simulated output in the presence of thermal noise has shown that an amplitude signal–to–noise ratio is needed of almost 10, in order to exploit all the possibilities of a resonant sphere (cfr [10]).

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APPENDIX A: DECOUPLING

In this appendix we decouple Eq. (4.21) in the dodecahedron arrangement.

Let us set $\boldsymbol{\eta}$ an 11–vector whose components are

$$\begin{aligned} \eta_i &= f_i \qquad && i \in \{1, \ldots, 5\} \\ \eta_{i+5} &= q_i \qquad && i \in \{1, \ldots, 6\} \end{aligned} \tag{A1}$$

With this position the Lagrangian of the system is written

$$L = \frac{1}{2} T_{ij} \dot{\eta}^i \dot{\eta}^j - \frac{1}{2} V_{ij} \eta^i \eta^j \tag{A2}$$

where

$$T_{ij} = M \begin{pmatrix} (1 + 3\alpha^2\mu/2\pi)\boldsymbol{\mathcal{I}}_5 & \alpha\mu\boldsymbol{p} \\ \alpha\mu\boldsymbol{p}^t & \mu\boldsymbol{\mathcal{I}}_6 \end{pmatrix} = M t_{ij} \tag{A3}$$

is the kinetic energy matrix, while

$$V_{ij} = M\omega_0^2 \begin{pmatrix} \boldsymbol{\mathcal{I}}_5 & 0 \\ 0 & \mu\boldsymbol{\mathcal{I}}_6 \end{pmatrix} = M\omega_0^2 v_{ij} \tag{A4}$$

is the potential energy matrix. In (A3) and (A4) we have put $\mu = m/M$. Motion equation is

$$T_{ij}\ddot{\eta}^j + V_{ij}\eta^j = 0 \tag{A5}$$

To decouple the motion, the eigenvalues and eigenvectors of the matrix

$$(V_{ij} - \omega^2 T_{ij}) = M\omega_0^2(v_{ij} - \lambda^2 t_{ij}) \tag{A6}$$

must be found (see [15, Cap. 6]). In (A6) we set $\lambda = \omega/\omega_0$. Putting equal to zero the determinant of matrix (A6) one obtains

$$\det(V_{ij} - \omega^2 T_{ij}) = (M\omega_0^2)^{11}\mu^6(1 - \lambda^2)\left[\lambda^4 - 2\left(1 + \frac{3\alpha^2}{4\pi}\mu\right)\lambda^2 + 1\right]^5 = 0 \tag{A7}$$

The three distinct eigenvalues are therefore

$$
\begin{aligned}
\lambda_{1,2,3,4,5} &= \lambda_- = 1 - \frac{1}{2}\sqrt{\frac{3\alpha^2}{2\pi}\mu} \\
\lambda_6 &= \lambda = 1 \\
\lambda_{7,8,9,10,11} &= \lambda_+ = 1 + \frac{1}{2}\sqrt{\frac{3\alpha^2}{2\pi}\mu}
\end{aligned}
\tag{A8}
$$

To look for eigenvectors of (A6) is the same as to find the matrix $\boldsymbol{A}$ which simultaneously diagonalizes the matrix of kinetic and potential energy $\boldsymbol{T}$ (A3) and $\boldsymbol{V}$ (A4). This matrix is found to be

$$
\boldsymbol{A} = \sqrt{\frac{1}{2M}} \begin{pmatrix} \lambda_- \boldsymbol{\mathcal{I}}_5 & \boldsymbol{0}_{(5\times 1)} & \lambda_+ \boldsymbol{\mathcal{I}}_5 \\ \sqrt{2\pi/3\mu}\lambda_- \boldsymbol{p}^t & \boldsymbol{1}_{(6\times 1)}/\sqrt{3\mu} & -\sqrt{2\pi/3\mu}\lambda_+ \boldsymbol{p}^t \end{pmatrix} \tag{A9}
$$

where we have introduced notation $\boldsymbol{k}_{(L\times N)}$ to indicate a matrix with dimensions $L \times N$ whose elements are all equal to k. $\boldsymbol{A}$ is also the transition matrix between the generalized coupled coordinates η_i to the normal coordinates ζ_i

$$
\eta_i = \sum_j A_{ij}\zeta_j \tag{A10}
$$

Let us now consider the forced system. Let $\boldsymbol{F} = (F_j)$ be the generalized forces corresponding to the generalized coordinates $\boldsymbol{\eta} = (\eta_j)$. Then, generalized forces $\boldsymbol{Q} = (Q_i)$ for the normal coordinates are

$$
\boldsymbol{Q} = \boldsymbol{A}^t \boldsymbol{F} \tag{A11}
$$

If the external forces act only on the sphere modes then (see (4.21))

$$
\begin{aligned}
F_i &= Mg_i & i &\in \{1,\ldots,5\} \\
F_i &= 0 & i &\in \{6,\ldots,11\}
\end{aligned}
\tag{A12}
$$

Therefore

$$
\boldsymbol{Q} = \sqrt{\frac{M}{2}} \begin{pmatrix} \lambda_- \boldsymbol{g} \\ 0 \\ \lambda_+ \boldsymbol{g} \end{pmatrix} \tag{A13}
$$

If we put

$$
\begin{aligned}
\boldsymbol{\zeta}_- &= (\zeta_i) & i &\in \{1,\ldots,5\} \\
\zeta &= \zeta_6 & & \\
\boldsymbol{\zeta}_+ &= (\zeta_i) & i &\in \{7,\ldots,11\}
\end{aligned}
\tag{A14}
$$

then eqs. (5.4) and (5.5) are immediately found.

APPENDIX B: CALCULATION OF $\boldsymbol{g}$

In this appendix we show how to find the components of vector $\boldsymbol{g}_n$ (see Sec. VI). First the amplitudes $R'_{nlm'}(t)$, defined in (2.8) must be calculated. With the given gravitational signal (6.3) one has that the only coefficients which does not vanish in O' are (cfr. with [3]):

$$
\begin{aligned}
R'_{n22}(t) &= -\sqrt{\frac{32\pi}{15}} R(\alpha_{n2} + 3\beta_{n2}) E_R(t) \\
R'_{n2-2}(t) &= -\sqrt{\frac{32\pi}{15}} R(\alpha_{n2} + 3\beta_{n2}) E_L(t) \\
R'_{n20}(t) &= -\sqrt{\frac{16\pi}{15}} R(\alpha_{n2} + 3\beta_{n2}) E_l(t)
\end{aligned}
\tag{B1}
$$

where, using the same notation as [3], we have

$$\alpha_{n2} = (MR)^{-1} \int a_{nl}(r)\rho r^3\, dr \qquad\qquad \beta_{n2} = M^{-1} \int b_{nl}(r)\rho r^2\, dr \tag{B2}$$

Now $G_{nml}(t)$ defined in (3.12), that is to say the coefficients of the gravitational wave decomposition in O are to be evaluated. Because of (B1) one has

$$G_{nml}(t) = \delta_{l2} \left(\mathcal{D}^{(2)}_{m2} R'_{n22}(t) + \mathcal{D}^{(2)}_{m-2} R'_{n2-2}(t) + \mathcal{D}^{(2)}_{m0} R'_{n20}(t) \right) \tag{B3}$$

Following [14,3] one has

$$\mathcal{D}^{(2)}_{mm'}(H, \delta) = \exp\,[imH] d^{(2)}_{mm'}(\delta + \pi/2) \tag{B4}$$

where $d^{(2)}_{mm'}(\beta)$ ($\beta = \delta + \pi/2$) are given by (see [14])

$$\boldsymbol{d}^{(2)}(\beta) = \begin{pmatrix} d_{22}(\beta) & -d_{12}(\beta) & d_{02}(\beta) & -d_{-12}(\beta) & d_{-22}(\beta) \\ d_{12}(\beta) & d_{11}(\beta) & d_{10}(\beta) & d_{1-1}(\beta) & -d_{-12}(\beta) \\ d_{02}(\beta) & -d_{10}(\beta) & d_{00}(\beta) & d_{10}(\beta) & d_{02}(\beta) \\ d_{-12}(\beta) & d_{1-1}(\beta) & -d_{10}(\beta) & d_{11}(\beta) & -d_{12}(\beta) \\ d_{-22}(\beta) & d_{-12}(\beta) & d_{02}(\beta) & d_{12}(\beta) & d_{22}(\beta) \end{pmatrix} \tag{B5}$$

where

$$\begin{aligned}
d_{22}(\delta + \pi/2) &= 3/8 - 1/2 \sin\delta - 1/8 \cos 2\delta \\
d_{-22}(\delta + \pi/2) &= 3/8 + 1/2 \sin\delta - 1/8 \cos 2\delta \\
d_{12}(\delta + \pi/2) &= -1/2 \cos\delta + 1/4 \sin 2\delta \\
d_{-12}(\delta + \pi/2) &= -1/2 \cos\delta - 1/4 \sin 2\delta \\
d_{11}(\delta + \pi/2) &= -1/2 \sin\delta - 1/2 \cos 2\delta \\
d_{1-1}(\delta + \pi/2) &= -1/2 \sin\delta + 1/2 \cos 2\delta \\
d_{02}(\delta + \pi/2) &= \sqrt{3/32}\ (1 + \cos 2\delta) \\
d_{10}(\delta + \pi/2) &= -\sqrt{3/8} \sin 2\delta \\
d_{00}(\delta + \pi/2) &= (1 - 3\cos 2\delta)/4
\end{aligned}$$

Setting $\gamma_n = -\sqrt{32\pi/15}\, R(\alpha_{n2} + 3\beta_{n2})$, the irreducible tensor of second rank $\boldsymbol{G}_n$, whose components are G_{n2m} is written

$$\boldsymbol{G}_n = \gamma_n \left[E_R(t) \begin{pmatrix} d_{22}e^{i2H} \\ d_{12}e^{iH} \\ d_{02} \\ d_{-12}e^{-iH} \\ d_{-22}e^{-i2H} \end{pmatrix} + E_L(t) \begin{pmatrix} d_{-22}e^{i2H} \\ -d_{-12}e^{iH} \\ d_{02} \\ -d_{12}e^{-iH} \\ d_{22}e^{-i2H} \end{pmatrix} + \frac{E_l(t)}{\sqrt{2}} \begin{pmatrix} d_{02}e^{i2H} \\ d_{10}e^{iH} \\ d_{00} \\ -d_{10}e^{-iH} \\ d_{02}e^{-i2H} \end{pmatrix} \right] \tag{B6}$$

Applying to $\boldsymbol{G}_n$ the operator $\boldsymbol{U}^{(2)}$ defined in (4.8) we get the real vector $\boldsymbol{g}_n$ whose components are given by (6.4).

APPENDIX C: INVERSION OF SYSTEM (6.4)

After manipulation of system (6.4) with definition (7.3) we find the following equation for $x = \tan H/2$:

$$c_{(0)} + 2\,c_{(1)}\,x + c_{(2)}\,x^2 + 4\,c_{(3)}\,x^3 - c_{(2)}\,x^4 + 2\,c_{(1)}\,x^5 - c_{(0)}\,x^6 = 0 \tag{C1}$$

where

$$\begin{aligned}
c_{(0)} &= a_2^2\,a_3 - \sqrt{3}\,a_1\,a_2\,a_4 - a_3\,{a_4}^2 + a_2\,a_4\,a_5 \\
c_{(1)} &= {a_2}^3 - 2\,a_2\,{a_3}^2 + \\
&\quad + \sqrt{3}\,a_1\,a_3\,a_4 + a_2\,{a_4}^2 + 2\,\sqrt{3}\,a_1\,a_2\,a_5 + 3\,a_3\,a_4\,a_5 - 2\,a_2\,{a_5}^2 \\
c_{(2)} &= -11\,{a_2}^2\,a_3 + 4\,{a_3}^3 + \\
&\quad + 7\,\sqrt{3}\,a_1\,a_2\,a_4 + 7\,a_3\,{a_4}^2 - 8\,\sqrt{3}\,a_1\,a_3\,a_5 - 15\,a_2\,a_4\,a_5 - 8\,a_3\,{a_5}^2 \\
c_{(3)} &= -{a_2}^3 + 4\,a_2\,{a_3}^2 - \\
&\quad - 3\,\sqrt{3}\,a_1\,a_3\,a_4 - 3\,a_2\,{a_4}^2 - 2\,\sqrt{3}\,a_1\,a_2\,a_5 - 5\,a_3\,a_4\,a_5 + 2\,a_2\,{a_5}^2
\end{aligned}$$

Declination angle δ is found from

$$\tan\delta = \frac{a_4\cos 2H - a_5\sin 2H}{a_2\cos H - a_3\sin H} \tag{C2}$$

while the three polarization amplitudes are given by

$$E_\times = (a_4\cos 2H - a_5\sin 2H)\sin\delta + (a_2\cos H - a_3\sin H)\cos\delta \tag{C3}$$

$$\left(E_+ - \sqrt{3}E_l\right) = 2(a_3\cos H + a_2\sin H)\sin 2\delta - (a_5\cos 2H + a_4\sin 2H - \sqrt{3}a_1)\cos 2\delta \tag{C4}$$

$$\left(E_+ + \frac{1}{\sqrt{3}}E_l\right) = a_5\cos 2H + a_4\sin 2H + \frac{a_1}{\sqrt{3}} \tag{C5}$$

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FIG. 1. Simulation over 1000 attempts of reconstructed position of a source located at hour angle $H = 1.0\,rad$ and declination $\delta = 0.7\,rad$. Amplitude signal to noise ratio is 10. The incoming gravitational wave is assumed linearly polarized with $E_\times/E_+ = 3/4$.

FIG. 2. The same as Fig. 1 with $H = 1.0\,rad$, $\delta = 1.3\,rad$ and $E_\times/E_+ = 4/3$.

TABLE I. Reconstruction over 1000 attempts of gravitational signals coming from sources placed at hour angle H and declination δ. $\Delta\Omega$ is the angular resolution in steradians; E is the amplitude of the incoming waves in units of the SNR; E_+ and $E_\times$ are the polarization amplitudes; E_l is the eventual longitudinal component of the signal, which is used as a veto against spurious events.

	First case		Second case	
	Original data	Reconstruction	Original data	Reconstruction
$\delta\,(rad)$	0.7	0.7 ± 0.1	1.3	1.3 ± 0.1
$H\,(rad)$	1.0	1.0 ± 0.15	1.0	1.0 ± 0.5
$\Delta\Omega\,(sterad)$		1×10^{-2}		1×10^{-2}
$E\,(SNR)$	10	10 ± 1	10	10 ± 1
E_+	8	8 ± 1.5	6	4 ± 5
$E_\times$	6	6 ± 2	8	6.0 ± 4.5
E_l/E	0	0.0 ± 0.1	0	0.0 ± 0.1

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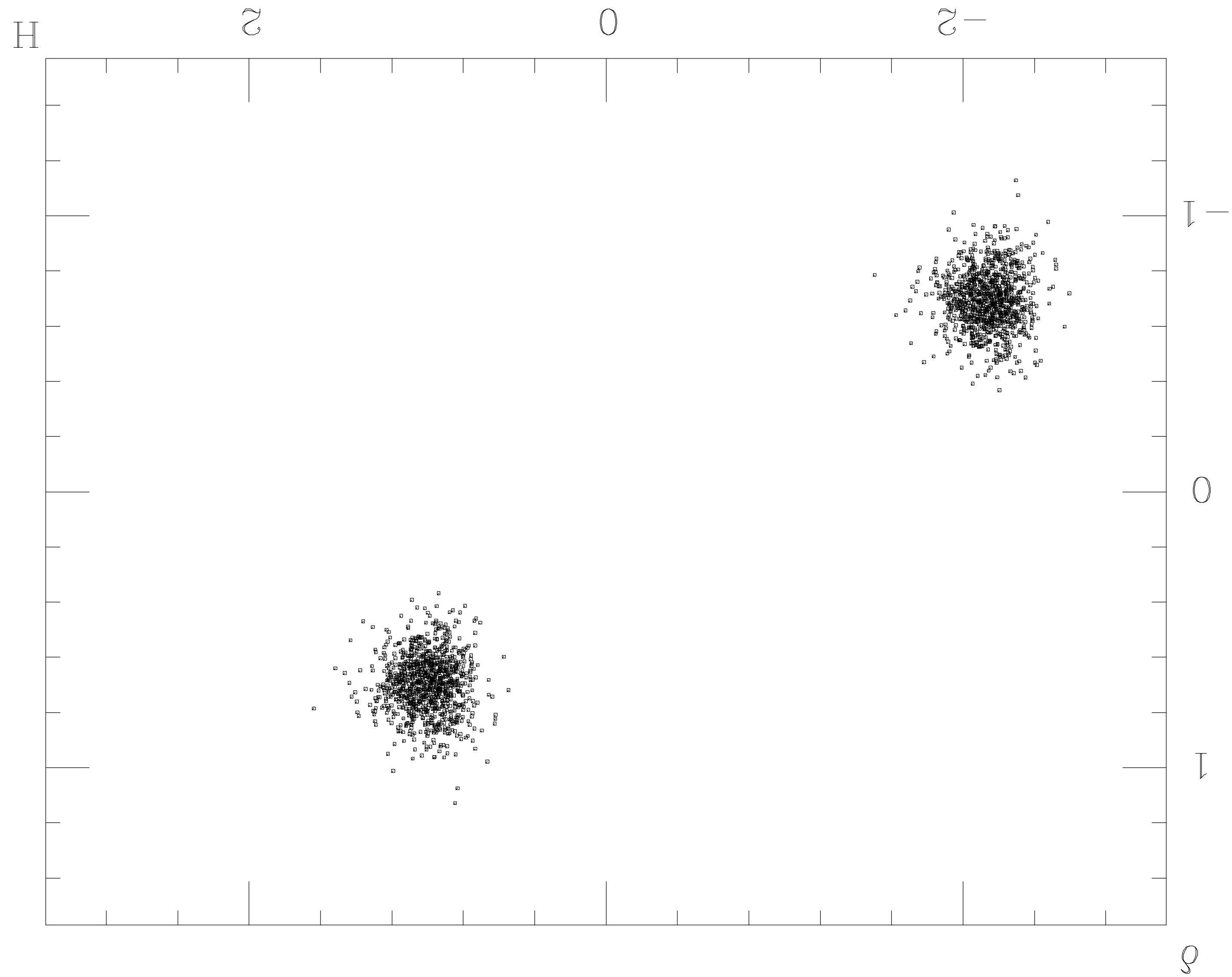

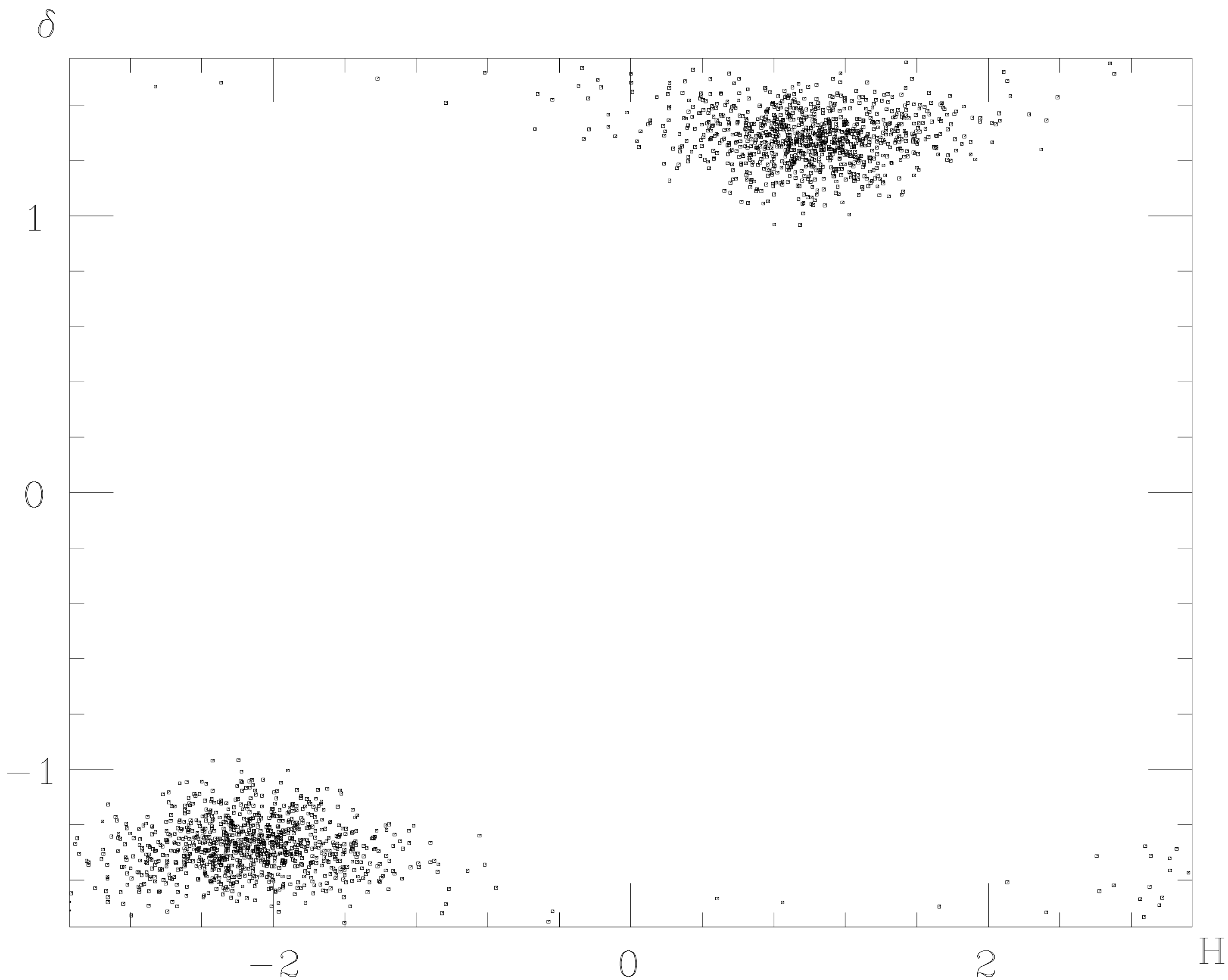
δ
1
0
−1
−2
0
2
H