ANGULAR MOMENTUM, QUATERNION, OCTONION, AND LIE-SUPER ALGEBRA OSP(1,2)*

by

Susumu Okubo

Department of Physics and Astronomy

University of Rochester, Rochester, NY 14627 U.S.A.

Abstract

We will derive both quaternion and octonion algebras as the Clebsch-Gordan algebras based upon the su(2) Lie algebra by considering angular momentum spaces of spin one and three. If we consider both spin 1 and $\frac{1}{2}$ states, then the same method will lead to the Lie-super algebra osp(1,2). Also, the quantum generalization of the method is discussed on the basis of the quantum group $\mathrm{su_q}(2)$.

* Paper dedicated to the 90th birthday of Professor Ta-You Wu. It will be published in the Proceedings of the Commemoration of Prof. Wu by World Scientific Publications.

1. Introduction

The Clebsch-Gordan coefficients of the angular momentum algebra in the quantum mechanics have a rich structure. We have shown in ref. 1 (referred to as I hereafter) that the Clebsch-Gordan recoupling of the spin 3 system will effectively lead to the octonion algebra. Let $\psi_j(m) \equiv |j, m>$ with $m = j, j-1, \ldots, -j$ be the standard eigenfunction for the angular momentum j with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Let j be a integer and introduce now a $2j+1$ dimensional algebra with the product defined by

$$\psi_j(m_1) \cdot \psi_j(m_2) = b_j \sum_{m_3} C \begin{pmatrix} j & j & j \\ m_1 & m_2 & m_3 \end{pmatrix} \psi_j(m_3) \tag{1.1}$$

for a constant b_j. Here, $C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Clebsch-Gordan coefficient for recoupling of two angular momentum states j_1 and j_2 into j_3. Note that we are restricting here for consideration only of $j_1 = j_2 = j_3 = j$. Since we have

$$C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2,m_3} C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{1.2a}$$

as well as

$$C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2-j_3} C \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} \quad , \tag{1.2b}$$

we may rewrite Eq. (1.1) as

$$\psi_j(m_1) \cdot \psi_j(m_2) = b_j C_j(m_1, m_2) \psi_j(m_1 + m_2) \tag{1.3}$$

which satisfies

$$\psi_j(m_1) \cdot \psi_j(m_2) = (-1)^j \psi_j(m_2) \cdot \psi_j(m_1) \quad , \tag{1.4}$$

where we have set

$$C_j(m_1, m_2) = C \begin{pmatrix} j & j & j \\ m_1 & m_2 & m_1 + m_2 \end{pmatrix} \tag{1.5}$$

for simplicity. Especially, Eq. (1.4) implies that the algebra is commutative for $j =$ even but anti-commutative for $j =$ odd. We may also introduce a bilinear (not sesqui-linear) form by

$$< \psi_j(m_1) | \psi_j(m_2) > = (2j+1)^{\frac{1}{2}} C \begin{pmatrix} j & j & 0 \\ m_1 & m_2 & 0 \end{pmatrix} = (-1)^{j-m_1} \delta_{m_1+m_2,0} \tag{1.6}$$

now for both integer as well as half-integer values of j. Then, we will have

$$< \psi_j(m_2)|\psi_j(m_1) > = (-1)^{2j} < \psi_j(m_1)|\psi_j(m_2) > \tag{1.7}$$

which is symmetric for integer j but anti-symmetric for half integer j. It also satisfies the associative trace condition for integer j, i.e.

$$< \psi_j(m_1) \cdot \psi_j(m_2)|\psi_j(m_3) > = < \psi_j(m_1)|\psi_j(m_2) \cdot \psi_j(m_3) > \tag{1.8}$$

as we may easily verify from properties of the Clebsch-Gordan coefficient. We have shown in I that the algebra defined by Eq. (1.3) give the Lie algebra su(2) for $j = 1$ and the 7-dimensional exceptional Malcev algebra[2)] for the case of $j = 3$. They are intimately connected with the quaternion and octonion algebras as follows. We will adjoin the spin 0 state $e_0 \equiv |j = 0, m = 0 >$ and modify Eq. (1.3) as

$$\psi_j(m_1) * \psi_j(m_2) = a_j < \psi_j(m_1)|\psi_j(m_2) > e_0 + b_j C_j(m_1, m_2)\psi_j(m_1 + m_2) \tag{1.9a}$$

where e_0 acts as the unit element, i.e.

$$e_0 * \psi_j(m) = \psi_j(m) * e_0 = \psi_j(m) \quad , \tag{1.9b}$$

$$e_0 * e_0 = e_0 \quad . \tag{1.9c}$$

We also set

$$< e_0|\psi_j(m) > = < \psi_j(m)|e_0 > = 0 \quad , \tag{1.10a}$$

$$< e_0|e_0 > = 1 \quad . \tag{1.10b}$$

If we choose the constant a_j in Eq. (1.9a) suitably, then the modified algebra now lead to the quaternion and octonion algebras, respectively for $j = 1$ and $j = 3$. The group-theoretical reasons behind these statements will be found in I. In this note, we will first explicitly demonstrate these facts in terms of identities among Clebsch-Gordan coefficients in section 2. If we consider a system consisting of $j = 1$ and $j = \frac{1}{2}$, then the resulting algebra turns out to be the Lie-super algebra[3)] osp(1,2). Similarly, we can generalize the octonion algebra into a super algebra by now using spins 0, 3, and $\frac{3}{2}$. Finally, we will make a comment on the possibility of constructing quantum quaternion and quantum octonion algebras by use of the quantum Clebsch-Gordan coefficients[4)] of the quantum group $\mathrm{su_q}(2)$.

2. Quaternion and Octonion Algebras

Let us first consider the simplest case of spin one. It is then easy to verify that the Clebsch-Gordan coefficients satisfy the special relation of form

$$C_1(m_1, m_2)C_1(m_1 + m_2, m_3) = \frac{1}{2}\{(-1)^{m_2}\delta_{m_2+m_3,0} - (-1)^{m_1}\delta_{m_1+m_3,0}\} \tag{2.1}$$

for any values 0, and ± 1 for m_1, m_2, and m_3. As the consequence, the product defined by Eq. (1.3) satisfies

$$\begin{aligned}(\psi_1(m_1) \cdot \psi_1(m_2)) \cdot \psi_1(m_3) =& \frac{1}{2}(b_1)^2\{< \psi_1(m_1)|\psi_1(m_3) > \psi_1(m_2) \\ & - < \psi_1(m_2)|\psi_1(m_3) > \psi_1(m_1)\} \quad .\end{aligned} \tag{2.2}$$

Writing 3 generic elements of the algebra as

$$x = \sum_{m=-1}^{1} \alpha(m)\psi_1(m) \tag{2.3a}$$

$$y = \sum_{m=-1}^{1} \beta(m)\psi_1(m) \tag{2.3b}$$

$$z = \sum_{m=-1}^{1} \gamma(m)\psi_1(m) \tag{2.3c}$$

for constants $\alpha(m)$, $\beta(m)$, and $\gamma(m)$, then we have

$$x \cdot y = -y \cdot x \quad , \tag{2.4a}$$

$$(x \cdot y) \cdot z = \frac{1}{2}(b_1)^2\{< x|z > y - < y|z > x\} \tag{2.4b}$$

when we multiply $\alpha(m_1)\beta(m_2)\gamma(m_3)$ to both sides of Eq. (2.2) and sum over m_1, m_2, and m_3. Cyclically interchanging $x \to y \to z \to x$, and adding all, it leads to

$$(x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y = 0 \tag{2.5}$$

where we used the symmetry condition $< x|y > = < y|x >$ because of Eq. (1.7). Together with $x \cdot y = -y \cdot x$ by Eq. (2.4a), this implies that the present algebra is indeed a Lie algebra. It is easy to see that it is isomorphic to the su(2) for $b_1 \neq 0$ when we identify

$$\psi_1(0) = -\frac{1}{\sqrt{2}} b_1 J_3 \quad , \quad \psi_1(\pm 1) = \frac{i b_1}{2} J_\pm \quad , \tag{2.6}$$

and write $x \cdot y = [x, y]$ which then leads to the familiar relation $[J_3, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = 2J_3$.

Let us next adjoin the unit element e_0 and consider the modified algebra Eq. (1.9) with

$$x * y = a_1 < x|y > e_0 + x \cdot y \quad . \tag{2.7}$$

We calculate then

$$(x * y) * z = a_1 < x|y > z + a_1 < x \cdot y|z > e_0 + (x \cdot y) \cdot z$$
$$x * (y * z) = a_1 < y|z > x + a_1 < x|y \cdot z > e_0 + x \cdot (y \cdot z)$$

so that

$$(x * y) * z - x * (y * z) = a_1 < x|y > z - a_1 < y|z > x + (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{2.8}$$

where we used $< x \cdot y|z > = < x|y \cdot z >$ by Eq. (1.8). However

$$\begin{aligned}(x \cdot y) \cdot z - x \cdot (y \cdot z) &= (x \cdot y) \cdot z + (y \cdot z) \cdot x = -(z \cdot x) \cdot y \\ &= -\frac{1}{2}(b_1)^2 \{< z|y > x - < x|y > z\}\end{aligned}$$

from Eqs. (2.4) and (2.5). Eq. (2.8) then leads to

$$(x * y) * z - x * (y * z) = \left[a_1 + \frac{1}{2}(b_1)^2\right] \{< x|y > z - < y|z > x\} \quad .$$

Therefore, if we choose the constant a_1 to be

$$a_1 = -\frac{1}{2}(b_1)^2 \tag{2.9}$$

we will find

$$(x * y) * z = x * (y * z) \tag{2.10}$$

so that the new product is associative . It is easy to verify that this together with the unit element e_0 leads to the quaternion algebra.

We will next consider the case of $j = 3$. Although we will no longer have such a simple relation as Eq. (2.1), the following identity can be verified to be valid:

$$\begin{aligned} &C_3(m_1, m_3)C_3(m_2, m_1 + m_3) + C_3(m_2, m_3)C_3(m_1, m_2 + m_3) \\ &\qquad = \frac{1}{6}\{2(-1)^{m_1}\delta_{m_1+m_2,0} - (-1)^{m_3}[\delta_{m_1+m_3,0} + \delta_{m_2+m_3,0}]\} \end{aligned} \tag{2.11}$$

for values of $m_1,\ m_2,\ m_3$ being $0,\ \pm1,\ \pm2,$ and ± 3. Setting

$$\begin{aligned} x &= \sum_{m=-3}^{3} \alpha(m)\psi_3(m) \quad , \\ y &= \sum_{m=-3}^{3} \beta(m)\psi_3(m) \quad , \\ z &= \sum_{m=-3}^{3} \gamma(m)\psi_3(m) \quad , \end{aligned} \tag{2.12}$$

Eq. (2.11) leads now to

$$(x \cdot z) \cdot y + (y \cdot z) \cdot x = \frac{1}{6}(b_3)^2\{2 < x|y > z - < x|z > y - < y|z > x\} \tag{2.13}$$

when we note Eqs. (1.3) and (1.6). As we remarked in I, Eq. (2.13) implies that it corresponds to the 7-dimensional simple Malcev algebra.[2)] Moreover, if we adjoin the unit element e_0 as in Eqs. (1.9) and (2.7), then it gives the octonion algebra, provided that we assign a suitable value of a_3. However, we will not go into detail here which can be found in I.

The case of $j = 2$ may also be of some interest, since we now have $x \cdot y = y \cdot x$. In that case, the Clebsch-Gordan coefficients satisfy

$$\begin{aligned} &C_2(m_1, m_2)C_2(m_3, m_1 + m_2) \\ &\quad + C_2(m_2, m_3)C_2(m_1, m_2 + m_3) + C_2(m_3, m_1)C_2(m_2, m_3 + m_1) \\ &= \frac{2}{7}\{(-1)^{m_1}\delta_{m_1+m_2,0} + (-1)^{m_2}\delta_{m_2+m_3,0} + (-1)^{m_3}\delta_{m_3+m_1,0}\} \end{aligned} \tag{2.14}$$

instead of Eq. (2.11) for values of $m_1,\ m_2,\ m_3$ being $0,\ \pm1,\ \pm2$. We will then have the cubic equation

$$x^3 = \frac{2}{7}(b_2)^2 < x|x > x \tag{2.15}$$

for generic element x where $x^3 = (x \cdot x) \cdot x = x \cdot (x \cdot x)$. If we adjoin the unit element e_0, then it will give a Jordan algebra. However, we will not discuss the details.

It is sometimes more convenient to use quantities associated with the Cartesian, rather than polar coordinates. For example, the spin one system may be labelled simply as a vector ϕ_μ for $\mu = 1, 2, 3$. Then, the product Eq. (1.1) will be simply written as

$$\phi_\mu \cdot \phi_\nu = b_1' \sum_{\lambda=1}^{3} \epsilon_{\mu\nu\lambda} \phi_\lambda \tag{2.16}$$

for another constant b_1', where $\epsilon_{\mu\nu\lambda}$ is the totally anti-symmetric Levi-Civita symbol in 3-dimension. Choosing $b_1' = 1$ and introducing the new product now by $\phi_\mu * \phi_\nu = -\delta_{\mu\nu} e_0 + \sum_{\lambda=1}^{3} \epsilon_{\mu\nu\lambda}\phi_\lambda$, it immediately gives the quaternion algebra. Similarly, the spin 3 system can be specified[5)] by the totally symmetric traceless tensor $\phi_{\mu\nu\lambda}(\mu, \nu, \lambda = 1, 2, 3)$ i.e.

$$\begin{aligned} &\text{(i)} \qquad && \phi_{\mu\nu\lambda} = \text{ symmetric in } \mu,\ \nu, \text{ and } \lambda \\ &\text{(ii)} \qquad && \sum_{\mu=1}^{3} \phi_{\mu\mu\lambda} = 0 \qquad (\lambda = 1, 2, 3) \quad . \end{aligned} \tag{2.17}$$

We now introduce the dot product by

$$\begin{aligned} \phi_{\mu\nu\lambda} \cdot \phi_{\alpha\beta\gamma} = & \frac{b}{3!3!} \sum_{P,P'} \sum_{\tau=1}^{3} \{ \delta_{\mu\alpha} \epsilon_{\nu\beta\tau} \phi_{\lambda\gamma\tau} \\ & - \frac{1}{5} \delta_{\mu\nu} \epsilon_{\lambda\beta\tau} \phi_{\alpha\gamma\tau} + \frac{1}{5} \delta_{\alpha\beta} \epsilon_{\gamma\nu\tau} \phi_{\mu\lambda\tau} \} \end{aligned} \tag{2.18}$$

for another constant b, where the summations on P and P' stand for 3! permutations of $\mu,\ \nu,\ \lambda$, and of $\alpha,\ \beta,\ \gamma$, respectively. Choosing $b = -5$, and identifying

$$\begin{aligned} & e_1 = -\sqrt{\frac{3}{2}} \phi_{233} \quad , && e_2 = 2\sqrt{\frac{3}{5}} \phi_{123} \quad , \\ & e_3 = \frac{1}{\sqrt{10}} (\phi_{222} - 3\phi_{112}) \quad , && e_4 = \sqrt{\frac{3}{2}} \phi_{133} \quad , \\ & e_5 = -\sqrt{\frac{3}{5}} (\phi_{311} - \phi_{322}) \quad , && e_6 = -\frac{1}{\sqrt{10}} (\phi_{111} - 3\phi_{122}) \quad , \\ & e_7 = \phi_{333} \quad , && \end{aligned} \tag{2.19}$$

we can verify that Eq. (2.18) is equivalent to

$$e_A \cdot e_B = \sum_{C=1}^{7} f_{ABC}\ e_C \tag{2.20}$$

for $A, B = 1, 2 \ldots, 7$ where f_{ABC} is totally anti-symmetric constants in A, B, C with values of 0, ± 1 as is tabulated in I. Then, adding the unit element $e \equiv e_0$, the algebra defined by

$$e_A * e_B = -\delta_{AB} e_0 + \sum_{C=1}^{7} f_{ABC}\ e_C \tag{2.21}$$

gives the standard octonion algebra.

3. Lie-super Algebra OSP(1,2)

If we consider algebras containing both integer and half-integer spin states, it will lead to super-algebras, where the integer spin states correspond to bosonic elements while the half-integer ones give the fermionic components. As a example, consider the system consisting of $j = 1$ and $j = \frac{1}{2}$, where we would have

$$\psi_1(M_1) \cdot \psi_1(M_2) = b_1 \sum_{M_3} C \begin{pmatrix} 1 & 1 & 1 \\ M_1 & M_2 & M_3 \end{pmatrix} \psi_1(M_3) \quad , \tag{3.1a}$$

$$\psi_{\frac{1}{2}}(m_1) \cdot \psi_{\frac{1}{2}}(m_2) = b_2 \sum_{M} C \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m_1 & m_2 & M \end{pmatrix} \psi_1(M) \quad , \tag{3.1b}$$

$$\psi_1(M_1) \cdot \psi_{\frac{1}{2}}(m_1) = a_1 \sum_{m_2} C \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ M_1 & m_1 & m_2 \end{pmatrix} \psi_{\frac{1}{2}}(m_2) \quad , \tag{3.1c}$$

$$\psi_{\frac{1}{2}}(m_1) \cdot \psi_1(M_1) = a_2 \sum_{m_2} C \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ m_1 & M_1 & m_2 \end{pmatrix} \psi_{\frac{1}{2}}(m_2) \quad , \tag{3.1d}$$

for some constants a_j and b_j. Note that Eqs. (3.1a) and (3.1b) imply

$$\psi_1(M_1) \cdot \psi_1(M_2) = -\psi_1(M_2) \cdot \psi_1(M_1) \tag{3.2a}$$

$$\psi_{\frac{1}{2}}(m_1) \cdot \psi_{\frac{1}{2}}(m_2) = \psi_{\frac{1}{2}}(m_2) \cdot \psi_{\frac{1}{2}}(m_1) \tag{3.2b}$$

because of Eq. (1.2b), while the commutability between $j = 1$ and $j = \frac{1}{2}$ components is not determined since the constants a_1 and a_2 are arbitrary. However, the symmetry strongly suggests the choice of $a_1 = a_2$ in Eqs. (3.1c) and (3.1d) so that we have

$$\psi_1(M_1) \cdot \psi_{\frac{1}{2}}(m_1) = -\psi_{\frac{1}{2}}(m_1) \cdot \psi_1(M_1) \quad . \tag{3.3}$$

Then, assigning the grade of 0 and 1 for $j = 1$ and $j = \frac{1}{2}$ components, respectively, it defines a super-algebra, since two generic elements x and y obey

$$x \cdot y = -(-1)^{xy} y \cdot x \tag{3.4}$$

in the standard convention where

$$(-1)^{xy} = \begin{cases} -1 & , \quad \text{if both } x \text{ and } y \text{ are fermionic, i.e. spin } \frac{1}{2} \\ +1 & , \quad \text{otherwise} \end{cases} \quad . \tag{3.5}$$

Moreover, if we choose the value of $a_1 = a_2$ suitably, then it can be verified to give a Lie-super algebra with the Jacobi identity

$$(-1)^{xz}(x \cdot y) \cdot z + (-1)^{yx}(y \cdot z) \cdot x + (-1)^{zy}(z \cdot x) \cdot y = 0 \quad . \tag{3.6}$$

Here, if we wish, we can use the more familiar notation of $[x, y]$ or $[x, y\}$ instead of $x \cdot y$. Further, the resulting Lie-super algebra corresponds to the ortho-symplectic one osp(1,2).

In order to prove these assertions made above, it is more convenient to use quantities in the Cartesian coordinate, where $\phi_\mu (\mu = 1, 2, 3)$ refers to spin 1 and the spinor $\xi_j (j = 1, 2)$ represents spin $\frac{1}{2}$. Since b_1 and b_2 in Eqs. (3.1a) and (3.1b) can always be suitably renormalized by adopting suitable normalizations for $\psi_1(M)$ and $\psi_{\frac{1}{2}}(m)$, the corresponding relations in the cartesian coordinate may be rewritten as

$$\phi_\mu \cdot \phi_\nu = i \sum_{\lambda=1}^{3} \epsilon_{\mu\nu\lambda} \phi_\lambda \quad , \quad (\mu, \nu = 1, 2, 3) \quad , \tag{3.7a}$$

$$\phi_\mu \cdot \xi_j = -\xi_j \cdot \phi_\mu = a' \sum_{k=1}^{2} \xi_k (\sigma_\mu)_{kj} \quad , \tag{3.7b}$$

$$\xi_j \cdot \xi_k = -\frac{i}{2} \sum_{\lambda=1}^{3} (\sigma_2 \sigma_\lambda)_{jk} \phi_\lambda \quad , \quad (j, k = 1, 2) \quad . \tag{3.7c}$$

Here, $\sigma_\mu (\mu = 1, 2, 3)$ are standards 2×2 Pauli matrices, and we note

$$(\sigma_2 \sigma_\lambda)^T = \sigma_2 \sigma_\lambda \quad , \quad \sigma_2^T = -\sigma_2$$

for the transposed matrix. The Jacobi identity Eq. (3.6) can be readily verified from Eqs. (3.7), if the constant a' in Eq. (3.7b) is chosen to be $a' = \frac{1}{2}$ which we assume hereafter. To show next that the Lie-super algebra is osp(1,2), we first rewrite Eq. (3.7a) by introducing $X_{ab}(a, b = 1, 2)$ by

$$X_{11} = -2(\phi_1 + i\phi_2) \quad , \quad X_{22} = 2(\phi_1 - i\phi_2)$$
$$X_{12} = X_{21} = -2\phi_3 \tag{3.8}$$

so that Eq. (3.7a) is rewritten as

$$\text{(i)} \qquad X_{ab} = X_{ba} \tag{3.9a}$$

$$\text{(ii)} \qquad X_{ab} \cdot X_{cd} = \epsilon_{bc} X_{ad} + \epsilon_{ac} X_{bd} + \epsilon_{bd} X_{ac} + \epsilon_{ad} X_{bc} \tag{3.9b}$$

for values of $a,\ b,\ c,\ d = 1, 2$ where we have set

$$\epsilon_{11} = \epsilon_{22} = 0 \qquad , \qquad \epsilon_{12} = -\epsilon_{21} = 1 \tag{3.10a}$$

and hence

$$\epsilon_{ab} = -\epsilon_{ba} \equiv (i\sigma_2)_{ab} \quad , \quad (a, b = 1, 2) \quad . \tag{3.10b}$$

Note that Eqs. (3.9) is the symplectic Lie algebra sp(2) which is isomorphic to su(2) by Eq. (3.8). Similarly, by setting

$$u_1 = -2\xi_1 \quad , \quad u_2 = 2\xi_2 \quad , \tag{3.11}$$

Eqs. (3.7b) is rewritten as

$$X_{ab} \cdot u_j = -u_j \cdot X_{ab} = \epsilon_{aj} u_b + \epsilon_{bj} u_a \tag{3.12}$$

if we choose $a' = \frac{1}{2}$. Finally, Eq. (3.7c) leads to

$$u_j \cdot u_k = X_{jk} \quad , \quad (j, k = 1, 2) \quad . \tag{3.13}$$

Now, we add an extra index 0 in addition to 1 and 2, and set

$$\begin{aligned} &X_{j0} = X_{0j} = u_j \quad , \quad (j = 1, 2) \\ &X_{00} = 0 \quad . \end{aligned} \tag{3.14}$$

Then Eqs. (3.9), (3.12), and (3.13) are rewritten as

$$\begin{aligned} X_{AB} \cdot X_{CD} = &\epsilon_{BC} X_{AD} + (-1)^{B \cdot C} \epsilon_{AC} X_{BD} \\ &+ (-1)^{B \cdot C} \epsilon_{BD} X_{AC} + (-1)^{A \cdot (B+C)} \epsilon_{AD} X_{BC} \end{aligned} \tag{3.15}$$

for $A,\ B,\ C,\ D = 0, 1,$ and 2. Here, we have set

$$\epsilon_{AB} = \begin{cases} \epsilon_{ab} & \text{if } A = a \text{ and } B = b \\ 1 & \text{if } A = B = 0 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{3.16}$$

Especially, both ϵ_{AB} and X_{AB} satisfies the symmetry conditions

$$\epsilon_{AB} = -(-1)^{A \cdot B} \epsilon_{BA} \quad , \tag{3.17a}$$

$$X_{AB} = (-1)^{A \cdot B} X_{BA} \tag{3.17b}$$

where $(-1)^{A \cdot B}$ is defined by

$$(-1)^{A \cdot B} = \begin{cases} -1 \ , & \text{if } A = B = 0 \\ +1 & \text{otherwise} \end{cases} \tag{3.18}$$

since the index 0 corresponds to the fermionic variable while other ones 1 and 2 refer to the bosonic ones. The relation Eq. (3.15) with Eqs. (3.17) defines the Lie-super algebra osp(1,2) if we identify $x \cdot y = [x, y]$. In this connection, we simply mention the fact that Lie-super algebra osp(n,m) is intimately related to para-statistics[6)] where the boson and fermion operators do no longer commute with each other.

We will next introduce a bilinear form by

$$< \phi_\mu | \phi_\nu > = \delta_{\mu\nu} \quad , \quad (\mu, \nu = 1, 2, 3) \tag{3.19a}$$

$$< \xi_j | \xi_k > = i(\sigma_2)_{jk} = \epsilon_{jk} \quad , \quad (j, k = 1, 2) \tag{3.19b}$$

$$< \xi_j | \phi_\mu > = < \phi_\mu | \xi_j > = 0 \quad . \tag{3.19c}$$

Then, it satisfies

$$\text{(i)} \quad < x | y > = 0 \ , \ \text{unless } x \text{ and } y \text{ are both bosonic or fermionic} \tag{3.20a}$$

$$\text{(ii)} \quad < y | x > = (-1)^{xy} < x | y > \tag{3.20b}$$

$$\text{(iii)} \quad < x \cdot y | z > = < x | y \cdot z > \tag{3.20c}$$

$$\text{(iv)} \quad < x | y > \ \text{is non} - \text{degenerate} \tag{3.20d}$$

so that $< \cdot | \cdot >$ is a supersymmetric bilinear non-degenerate associative form.

We now adjoin the unit element e_0 and define a new product by

$$x * y = - < x|y > e_0 - ix \cdot y \tag{3.21a}$$

$$x * e_0 = e_0 * x = x \quad , \quad e_0 * e_0 = e_0 \quad . \tag{3.21b}$$

We see then that the 4 bosonic elements e_0, ϕ_1, ϕ_2, and ϕ_3 define the usual quaternion algebra. Therefore, Eqs. (3.21) may be regarded as a super generalization of the quaternion algebra. However, it is no longer associative when the product involves fermionic element. We can moreover prove that the algebra is super-quadratic, super-flexible, and super-Lie-admissible, although we will not go into detail.

We can repeat a similar analysis for octonion algebra. We now consider a system consisting of $j = 0$, $j = 3$, and $j = \frac{3}{2}$. For products involving $j = \frac{3}{2}$, the corresponding Clebsch-Gordan algebra will be given by

$$\psi_{\frac{3}{2}}(m_1) \cdot \psi_{\frac{3}{2}}(m_2) = a_1 \sum_M C \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 3 \\ m_1 & m_2 & M \end{pmatrix} \psi_3(M) \tag{3.22a}$$

$$\begin{aligned} \psi_{\frac{3}{2}}(m_1) \cdot \psi_3(M_1) &= -\psi_3(M_1) \cdot \psi_{\frac{3}{2}}(m_1) \\ &= a_2 \sum_{m_2} C \begin{pmatrix} \frac{3}{2} & 3 & \frac{3}{2} \\ m_1 & M_1 & m_2 \end{pmatrix} \psi_{\frac{3}{2}}(m_2) \end{aligned} \tag{3.22b}$$

for some constants a_1 and a_2. Then, a similar construction gives the octonion algebra for the bosonic space, and the algebra may be considered also as a super generalization of the octonion algebra. However, we will not go into its detail.

4. Quantum Clebsch-Gordan Algebra

The idea explained in the previous sections can be extended for any quantum group L. Consider the quantum group $\mathrm{su_q}(2)$ which is defined[4)] by the commutation relations

$$[H, J_\pm] = \pm J_\pm \tag{4.1a}$$

$$[J_+, J_-] = \frac{t^{2H} - t^{-2H}}{t - t^{-1}} \tag{4.1b}$$

for a constant parameter $t(= q^{\frac{1}{2}})$. The co-product $\Delta \ : \ L \to L \otimes L$ is specified by

$$\Delta(J_\pm) = t^{-H} \otimes J_\pm + J_\pm \otimes t^H \quad , \tag{4.2a}$$

$$\Delta(H) = 1 \otimes H + H \otimes 1 \tag{4.2b}$$

which satisfies

$$\Delta([x,y]) = [\Delta(x), \Delta(y)] \quad . \tag{4.3}$$

Moreover, the anti-pode $S \ : \ L \to L$ operates as

$$S(t^{\pm 2H}) = t^{\mp 2H} \quad , \quad S(J_\pm) = -t^{\pm 2} J_\pm \tag{4.4}$$

which obeys anti-morphism relation

$$S(xy) = S(y)S(x) \quad . \tag{4.5}$$

Finally, the co-unit ϵ is given by

$$\epsilon(t^{\pm 2H}) = 1 \quad , \quad \epsilon(J_\pm) = 0 \quad . \tag{4.6}$$

These operations define the Hopf algebra, i.e.

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad , \tag{4.7a}$$

$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id \quad , \tag{4.7b}$$

$$\epsilon \circ S = \epsilon \quad , \tag{4.7c}$$

$$\sigma \circ (S \otimes S) \circ \Delta = \Delta \circ S \tag{4.7d}$$

where σ in Eq. (4.7d) stands for the permutation operation.

Let $|j, m>_q$ now be the representation of $\mathrm{su_q}(2)$ with[4)]

$$H|j,m>_q = m|j,m>_q \tag{4.8a}$$

$$J_\pm|j,m>_q = ([j \mp m]_q [j \pm m + 1]_q)^{\frac{1}{2}} |j, m \pm 1>_q \tag{4.8b}$$

where

$$[n]_q = \frac{t^n - t^{-n}}{t - t^{-1}} = t^{(n-1)} + t^{(n-3)} + \ldots + t^{-(n-1)} \tag{4.9}$$

for non-negative integer n. Then, the quantum Clebsch-Gordan algebra for the integer angular momentum state j will be given by

$$|j, m_1>_q \cdot |j, m_2>_q = b_j \sum_{m_3} C_q \begin{pmatrix} j & j & j \\ m_1 & m_2 & m_3 \end{pmatrix} |j, m_3>_q \tag{4.10}$$

for the quantum Clebsch-Gordan coefficient[4),7)] $C_q \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$. This replaces Eq. (1.1). The product defined by Eq. (4.10) behaves covariantly under actions of $\mathrm{su_q}(2)$ in the following sense. Let $m\ :\ V\otimes V \to V$ be the multiplication operation in the $2j+1$ dimensional representation space V, i.e.

$$m(x\otimes y) = x\cdot y \quad , \quad x,\ y\ \epsilon\ V \quad . \tag{4.11a}$$

Then, by the construction of the quantum Clebsch-Gordan coefficients, it must satisfy the relation

$$g\circ m = m\circ\Delta(g) \tag{4.11b}$$

for any $g\ \epsilon\ \mathrm{su_q}(2)$. In order to illustrate that Eq. (4.11b) is the statement of covariance, let us consider the case of the ordinary su(2) Lie algebra corresponding to the choice $t=1$. Then,

$$\Delta(g) = g\otimes 1 + 1\otimes g$$

so that Eq. (4.11b) operated to $x\otimes y$ will reproduce the standard formula

$$g(x\cdot y) = (gx)\cdot y + x\cdot(gy)$$

for the action of the Lie algebra as a derivation.

In what follows, we will restrict ourselves to the special case of $j=1$ and set

$$x_0 = |1,0>_q \quad , \tag{4.12a}$$

$$x_\pm = |1,\pm 1>_q \quad . \tag{4.12b}$$

Then, Eq. (4.10) will lead to the multiplication table of

$$x_0\cdot x_0 = \beta(t-t^{-1})x_0 \quad , \tag{4.13a}$$

$$x_0\cdot x_\pm = \mp\beta t^{\mp 1}x_\pm \quad , \tag{4.13b}$$

$$x_\pm\cdot x_0 = \pm\beta t^{\pm 1}x_\pm \quad , \tag{4.13c}$$

$$x_\pm\cdot x_\mp = \pm\beta x_0 \quad , \tag{4.13d}$$

$$x_\pm\cdot x_\pm = 0 \tag{4.13e}$$

for a suitable normalization constant β, which satisfies Eq. (4.11b). For $t = 1$, this reproduces the results of section 2. Note that the algebra given by Eqs. (4.13) is no longer anti-commutative. It still possesses a involution operation ω defined by

$$\omega(x_0) = x_0 \quad , \quad \omega(x_\pm) = x_\mp \tag{4.14}$$

which satisfies

$$\omega(x \cdot y) = \omega(y) \cdot \omega(x) \quad . \tag{4.15}$$

For simplicity, we will normalize the constant β to be $\beta = 1$ in what follows. Then, the associator given by

$$(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{4.16}$$

can be verified to satisfy

$$(x, y, z) = B(x, y)z - B(y, z)x \tag{4.17}$$

where the bilinear form $B(x, y)$ is defined by

$$\begin{aligned} &B(x_+, x_-) = t \quad , \quad B(x_-, x_+) = \frac{1}{t} \quad , \\ &B(x_0, x_0) = -1 \quad , \quad B(x_\pm, x_0) = B(x_0, x_\pm) = 0 \quad . \end{aligned} \tag{4.18}$$

As a matter of fact, we have

$$B(x_{m_1}, x_{m_2}) = \text{ constant } C_q \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} \quad . \tag{4.19}$$

We also note that $B(x, y)$ is no longer symmetric but is associative, i.e.

$$B(x \cdot y, z) = B(x, y \cdot z) \quad . \tag{4.20}$$

Moreover, it satisfies

$$B(\omega(x), \omega(y)) = B(y, x) \tag{4.21}$$

for the involution ω given by Eq. (4.14).

From Eq. (4.17), we see that the algebra is not flexible, but is Lie-admissible[1)] since it obeys

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad . \tag{4.22}$$

If we now adjoin the unit element e_0 with the new product * by

$$x * y = x \cdot y - B(x,y)e_0 \quad , \tag{4.23}$$

it is easy to verify from these equations that it is associative, i.e.

$$(x * y) * z = x * (y * z) \quad . \tag{4.24}$$

Actually, the new algebra is isomorphic to the quaternion algebra so that ththe quantum quaternion algebra is nothing but the same as the usual quaternion algebra.

We can apply the same method for systems involving both $j = 1$ and $\frac{1}{2}$ states to obtain a quantum-deformed super algebra of osp(1,2). Analogously, if we consider $j = 3$, then it will lead to a quantum generalization of the octonion algebra. However, by a reason not given here, we have to actually use the quantum deformation of the 7-dimensional representation of the exceptional Lie algebra G_2 rather than the $j = 3$ states of su(2) in order to properly describe the quantum octonion algebra. These, however, will be studied in the future.

Acknowledgement

The present article is dedicated to the 90th birthday of Professor Ta-You Wu of Tsin-Hua and Jiao Tong Universities. It is also supported in part by the U.S. Department of Energy Grant No. DE-FG02-91ER40685.

References

1. S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics* (Cambridge University Press, Cambridge, 1995).
2. H.C. Myung, *Malcev-Admissible Algebras* (Birkhäuser, Boston, 1986).
3. M. Scheunert, *The Theory of Lie Super Algebras* (Springer-Verlag, Berlin, 1979).
4. L.C. Biedenharn and M.A. Lohe, *Quantum Group Symmetry and q-Tensor Algebras* (World Scientific, Singapore, 1995).
5. H. Weyl, *The Classical Groups* (Princeton University, Princeton 1939).
6. S. Okubo, Jour. Math. Phys. **35**, 2785 (1994).
7. A.N. Kirillov and N.Yu. Reshitikhin, USSR Academy of Sciences (unpublished) 1988, L. Vaksman, Sov. Math. Dokl. **39**, 467 (1989).