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Quantum effects in the Alcubierre warp drive spacetime

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Abstract

The expectation value of the stress-energy tensor of a free conformally invariant scalar field is computed in a two-dimensional reduction of the Alcubierre "warp drive" spacetime. The stress-energy is found to diverge if the apparent velocity of the spaceship exceeds the speed of light. If such behavior occurs in four dimensions, then it appears implausible that "warp drive" behavior in a spacetime could be engineered, even by an arbitrarily advanced civilization.

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Alcubierre [1] has described a spacetime which has several of the properties associated with the "warp drive" of science fiction. By causing the spacetime to contract in front of a spaceship, and expand behind, the Alcubierre warp drive spacetime allows a spaceship to have an apparent speed relative to distant objects much greater than the speed of light.

The stress-energy needed to have a spacetime of this sort is known to require matter which violates the weak, strong, and dominant energy conditions [1]. While quantized fields can locally violate the energy conditions, Pfenning and Ford [2] have recently demonstrated that the configuration of exotic matter needed to generate the warp "bubble" around the spaceship is quite implausible.

In this letter, a different issue involving quantum effects and the warp drive spacetime is examined. The curved spacetime associated with the warp drive will create a nonzero expectation value for the stress-energy of a quantized field in that spacetime. This field is assumed to be a spectator in the spacetime, not responsible for the stress-energy which supports the exotic warp drive metric. While calculating the expectation value of the stress-energy of a quantized field in a spacetime is generally an extremely difficult task, the work involved is greatly reduced if one confines attention to a two-dimensional spacetime. The warp drive spacetime admits a natural two-dimensional reduction containing the worldline of the spaceship. A coordinate transformation then renders the two-dimensional metric into a static form. For a conformally invariant massless quantized scalar field, the stress-energy is then completely determined by the trace anomaly, conservation, and the values of two integration constants which are determined by the state of the field [3,4].

The resulting expressions for $\langle T_\mu{}^\nu \rangle$ are found to be everywhere regular so long as the ship does not exceed the speed of light, $v < 1$. However, for apparent ship velocities exceeding the speed of light, the stress-energy diverges at a particular distance from the ship dependent upon the speed. This divergence is associated with an event horizon which forms in the two-dimensional spacetime. If the instability is not an artifact of working in two dimensions, then the spaceship would presumably be precluded from attaining apparent velocities greater than light due to metric backreaction effects.

The warp drive metric proposed by Alcubierre may be written as

$$ds^2 = -dt^2 + (dx - vf(r)dt)^2 + dy^2 + dz^2, \tag{1}$$

where v is the apparent velocity of the spaceship,

$$v = \frac{dx_s(t)}{dt}, \tag{2}$$

$x_s(t)$ is the trajectory of the spaceship (chosen to be along the x direction), r is defined by

$$r = \left[(x - x_s(t))^2 + y^2 + z^2\right]^{1/2}, \tag{3}$$

and f is an arbitrary function which decreases from unity at $r = 0$ (the location of the spaceship) to zero at infinity. Alcubierre gave a particular example of such a function,

$$f_A(r) = \frac{\tanh(\sigma(r + R)) - \tanh(\sigma(r - R))}{2\tanh(\sigma R)}, \tag{4}$$

where σ and R are positive arbitrary constants.

In this letter, the function f will not be constrained to the particular choice made by Alcubierre; f may be chosen arbitrarily subject only to the boundary conditions at $r = 0$ and infinity. In order to simplify the analysis of the effects of the spacetime on the quantized field, the velocity of the spaceship will be taken to be constant, $v = v_0$, which then implies that

$$x_s(t) = v_0 t, \tag{5}$$

and hence

$$r = \left[(x - v_0 t)^2 + y^2 + z^2\right]^{1/2}. \tag{6}$$

While the warp drive spacetime is not spherically symmetric, there is an obvious way to reduce the spacetime to two dimensions. The spacetime is cylindrically symmetric about the axis $y = z = 0$. The two-dimensional spacetime which includes the symmetry axis also contains the entire world line of the spaceship. The two-dimensional metric is then

$$ds^2 = -(1 - {v_0}^2 f^2)dt^2 - 2v_0 f dt dx + dx^2. \tag{7}$$

After setting $y = z = 0$, r reduces to

$$r = \sqrt{(x - v_0 t)^2}. \tag{8}$$

If attention is restricted to the half of the spacetime to the past of the spaceship ($x > v_0 t$), then the square root in Eq.(8) may be taken, so that in this domain, $r = x - v_0 t$ (results for the other half-space may be obtained by a trivial transformation).

Since the spaceship is traveling with constant velocity, there should exist a Lorentz-like transformation to a frame in which the ship is at rest. The required transformation is most easily understood if broken into several steps. First, since the metric components only depend on the quantity r, it is natural and possible to adopt this as a coordinate, transforming from (t, x) coordinates to (t, r) coordinates by making the replacement $dx = dr + v_0 dt$ in the metric of Eq.(7). This yields

$$ds^2 = -A(r)\left(dt - \frac{v_0(1 - f(r))}{A(r)}dr\right)^2 + \frac{dr^2}{A(r)}, \tag{9}$$

where

$$A(r) = 1 - {v_0}^2(1 - f(r))^2. \tag{10}$$

Next, the metric is brought into a comoving, diagonal form by defining a new time coordinate,

$$d\tau = dt - \frac{v_0(1 - f(r))}{A(r)}dr, \tag{11}$$

which gives the metric form

$$ds^2 = -A(r)d\tau^2 + \frac{1}{A(r)}dr^2. \tag{12}$$

This form of the metric is manifestly static. The coordinates have an obvious interpretation in terms of the occupants of the spaceship, as τ is the ship's proper time (since $A(r) \to 1$ as $r \to 0$). On the other hand, the coordinates are not asymptotically normalized in the usual fashion; for large r, far from the spaceship, $A(r)$ approaches $1 - {v_0}^2$ rather than unity. This may be corrected by defining yet one more set of coordinates, (T, Y), such that

$$T = \sqrt{1 - {v_0}^2}\,\tau, \quad Y = \frac{r}{\sqrt{1 - {v_0}^2}}. \tag{13}$$

The combined coordinate transformations taking (t, x) into (T, Y) have the asymptotic form of a Lorentz transformation far from the spaceship, at large r (or, equivalently, Y). In this limit,

$$T = \gamma(t - v_0 x), \quad Y = \gamma(x - v_0 t), \tag{14}$$

where γ is the usual special relativistic factor, $\gamma = 1/\sqrt{1 - {v_0}^2}$. The transformations to T and Y will include a factor i when $v_0 > 1$. This is an obvious consequence of transforming to the comoving frame when the apparent velocity exceeds unity. While there are no real complications associated with this transformation, the worry of even possibly having to deal with complex quantities will be avoided by using the (τ, r) coordinate system rather than the (T, Y) system.

Examining the form of the metric of Eq.(9), the coordinate system is seen to be valid for all $r > 0$ if $v_0 < 1$. If $v_0 > 1$, then there is a coordinate singularity (and event horizon) at the location r_0 such that $A(r_0) = 0$, or,

$$f(r_0) = 1 - \frac{1}{v_0}. \tag{15}$$

In this case ($v_0 > 1$), the spacetime is somewhat like DeSitter space. There exists an event horizon such that the static region of the spacetime is inside the horizon ($r < r_0$), and the horizon first appears at infinity and moves inward as the metric's adjustable parameter (v_0 or the cosmological constant, Λ) is increased.

The determination of the stress-energy tensor for a quantized conformally invariant scalar field in the spacetime of Eq.(9) is now straightforward [4]. Integration of the conservation equation and knowledge of the trace anomaly quickly gives

$$T_\tau{}^r = C_1, \tag{16}$$

$$T_\alpha{}^\alpha = -\frac{A''}{24\pi}, \tag{17}$$

$$T_r{}^r = \frac{C_2 + [A'(r_0)]^2}{96\pi A(r)} - \frac{(A')^2}{96\pi A(r)}, \tag{18}$$

where a prime denotes differentiation with respect to r and expectation value brackets have been suppressed for notational simplicity. The remaining components are trivially related to those given above, $T_\tau{}^\tau = T_\alpha{}^\alpha - T_r{}^r$, and $T_r{}^\tau = -C_1/A^2$. The integration constants C_1, C_2, and $A'(r_0)$ are determined by the choice of quantum state for the field.

If the field is assumed to be in a time independent and asymptotically empty state (the usual Minkowski vacuum state) at large r, so that

$$\lim_{r\to\infty} \langle T_\mu{}^\nu \rangle = 0, \tag{19}$$

then, since $A(r) \to 1 - {v_0}^2$ and $A'(r) \to 0$ as $r \to \infty$, this requires that

$$C_1 = C_2 + [A'(r_0)]^2 = 0. \tag{20}$$

With this choice of state, only the diagonal components of the stress-energy are nonzero. They take on the simple forms:

$$T_r{}^r = -\frac{(A')^2}{96\pi A(r)}, \tag{21}$$

$$T_\tau{}^\tau = -\frac{A''}{24\pi} + \frac{(A')^2}{96\pi A(r)}. \tag{22}$$

If $v_0 < 1$, then the function $A(r)$ is everywhere bounded and positive, and hence the (τ, r) coordinate system is regular. Examination of $T_\mu{}^\nu$ as given in Eqs.(21,22) shows that the components are everywhere finite.

If $v_0 > 1$, then there is an event horizon in the spacetime where $A(r_0) = 0$; the (τ, r) coordinate system suffers a coordinate singularity there. In order to determine the regularity of $\langle T_\mu{}^\nu \rangle$, it is necessary to evaluate the components in a frame regular at the horizon. There are several different ways this may be accomplished. The original (t, x) coordinate system is regular across the horizon. Unfortunately, however, the expressions for the components of $\langle T_\mu{}^\nu \rangle$ are long, complicated, and not particularly illuminating in this coordinate system. Alternately, one may evaluate the stress-energy components in an orthonormal frame attached to a freely falling observer. The procedure described in Ref. [4] may be followed to set up such a frame is in the static metric of Eq.(9). Near the horizon, the observed energy density will be proportional to

$$\langle \rho \rangle \sim \frac{T_r{}^r - T_\tau{}^\tau}{A(r)} = \frac{-A''}{24\pi A} - \frac{(A')^2}{48\pi A^2}. \tag{23}$$

Expanding Eq.(23) near the horizon, and expressing the result in terms of the original function f, yields

$$\langle \rho \rangle \sim \frac{-(f')^2}{48\pi} \left[f - \left(1 - \frac{1}{v_0} \right) \right]^{-2} + \ldots \quad , \tag{24}$$

where the ellipsis denotes less divergent terms. Clearly, there is no choice of function f which will cause the leading term in Eq.(24) to be finite as $f \to 1 - 1/v_0$.

This divergence has a simple origin. The event horizon which forms when the ship's velocity exceeds unity has a natural temperature associated with it,

$$T_{Hawking} = \frac{\kappa}{2\pi} = \frac{A'(r_0)}{4\pi} = v_0 \frac{f'(r_0)}{2\pi}. \tag{25}$$

If the quantum state is chosen to be asymptotically empty (essentially the Boulware vacuum state), then the temperature of the surrounding universe does not match the natural temperature of the black hole. It is then inevitable that the stress-energy of a quantized field will diverge on the horizon.

Since the "warp drive" spacetime is assumed to be associated with an intelligent engineering effort rather than an astrophysical cause, it is legitimate to ask whether the divergence might be "engineered" away. Presumably the spaceship designers and engineers could, for example, control the shape of the function $f(r)$. However, examination of Eq.(24) shows that the stress-energy of the quantized field will diverge on the event horizon regardless of the form of f. In a self-consistent solution of the semiclassical Einstein equations, the backreaction to this divergence would presumably prevent the spaceship from achieving an apparent velocity exceeding the speed of light.

The divergence on the horizon occurs because natural quantum state of the field, the Boulware vacuum, is not regular on the horizon. Warp drive designers might seek to have the spaceship modulate the quantized field in such a manner that it would locally, near the horizon, appear to be in a state which is regular there. They might eject particles or otherwise manipulate the field to simulate the Hartle-Hawking state at the appropriate temperature or the Unruh state near the horizon. However, since the horizon first appears at an infinite distance when $v = 1$ and subsequently moves inward, it is difficult to see how the state of the quantized field (presumably of *all* massless fields in nature) could be manipulated at such great distances from the ship.

Finally, one might object that the divergence perhaps only occurs along the single spatial direction in which the ship is traveling, since that is the only direction included in this two-dimensional calculation. A full four-dimensional calculation would be needed to settle this issue definitively.

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REFERENCES

[1] Alcubierre M. 1994 *Class. Quant. Grav.* **11** L73.

[2] Pfenning M. J. and Ford L. H. *Class. Quant. Grav.* 1997 **14** 1743.

[3] Davies P. C. W. , Fulling S. A. , and Unruh W. G. 1976 *Phys. Rev.* D**13** 2720.

[4] Loranz D. J. , Hiscock W. A. and Anderson, P. R. 1995 *Phys. Rev.* D**52** 4554.